

Bemerkungen zu

# Schleichenden Strömungen:

Trägheitskräfte

$\ll 1$

Reibkräfte

-  $\vec{v}$  klein

oder

-  $z$  sehr groß

oder

-  $h/c$  sehr klein

oder

-  $s$  sehr klein

• Bewegg. eines Kugel 1851 (Stokes)

1940 (Oseen)

• Strög. zw. parallelen

Platten mit  $h/c \ll 1$

1898 (Hele, Shaw)

• Praxis: Luft od. Wasser durch Sand

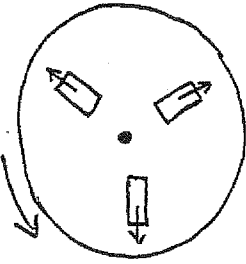
→ Grundwasserströmung

# Wirbelströmungen

Strömungen

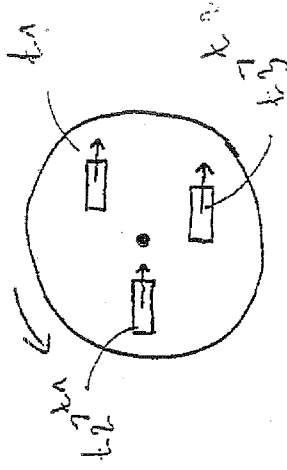


drehungsbehaftet



Wirbelströmungen

Potentialströme



Begriffe od. Wirbelströmungen

Dreh- od. Wirbelvektor  $\vec{\omega}$

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

$\vec{\omega}$  zeigt in Richtg. od. Drehachse d. Flüssigkeitselementen

Wirbellinien: Kurven entlang tangential zum Wirbelvektor

Wirbelröhre:

Wirbellinien d. Mantelfläche d. Wirbelfadens

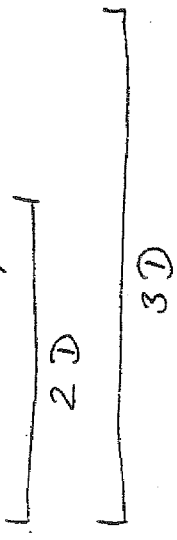
Richtung d. Wirbellinie

aus  $\vec{\omega} \parallel d\vec{s} \Rightarrow$

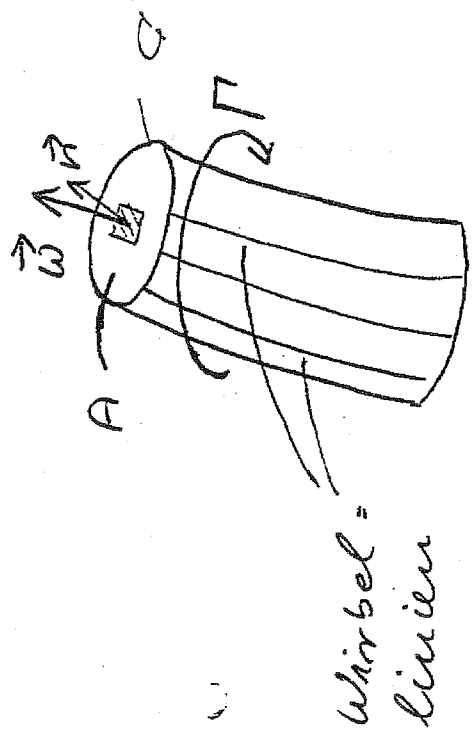
$$(\omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}) \times (\omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}) = \vec{0}$$

bzw.

$$\frac{\omega_x}{\omega_y} = \frac{dy}{\omega_z} = \frac{dz}{\omega_x}$$



$\int d\vec{s}$



Wirbelstrom odt. - fluß:

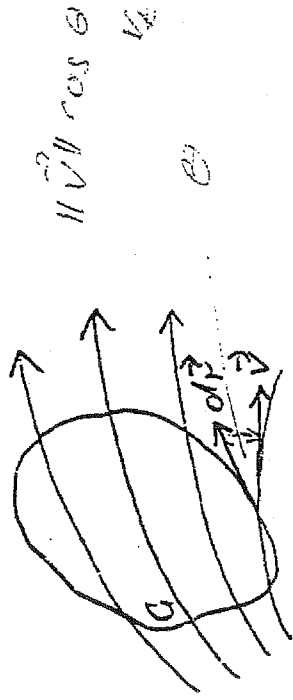
$$\Omega = \int_A \vec{\omega} \cdot \vec{n} \, dA$$

Wirbelfaden: Bündelg. aller durch A gehen: den Wirbellinien

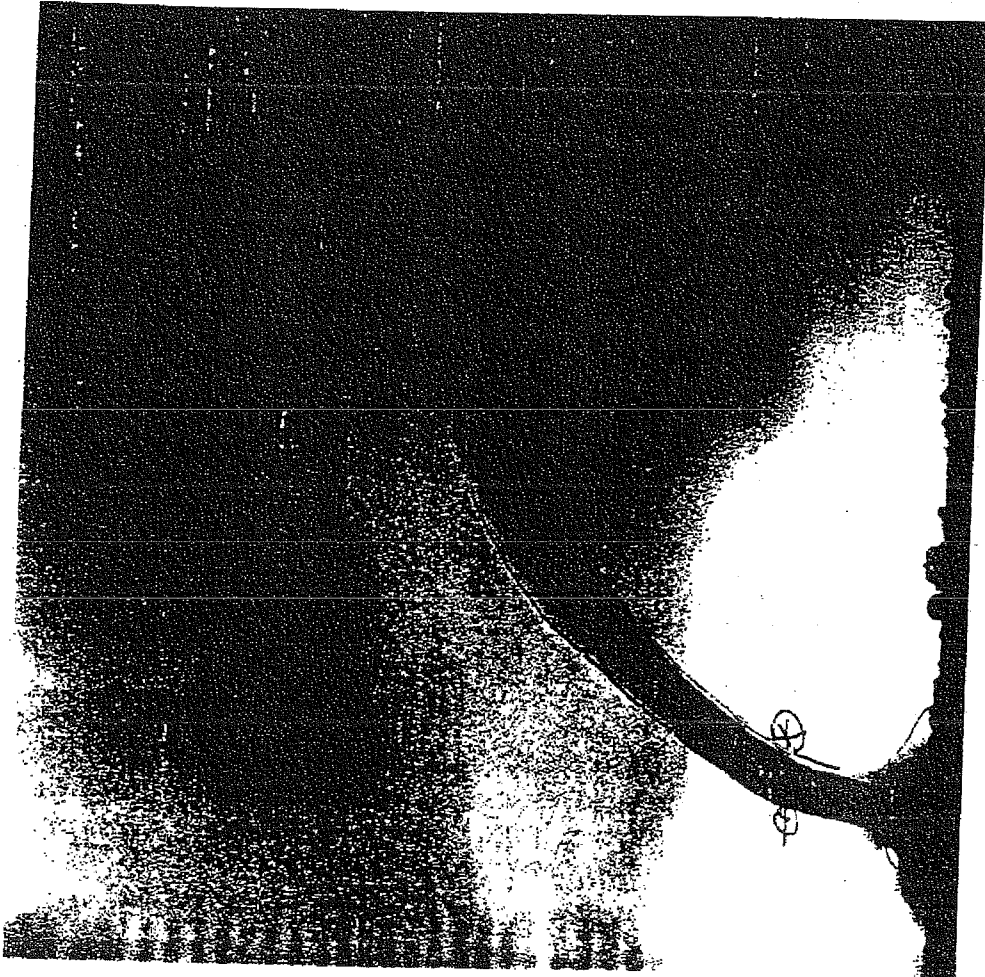
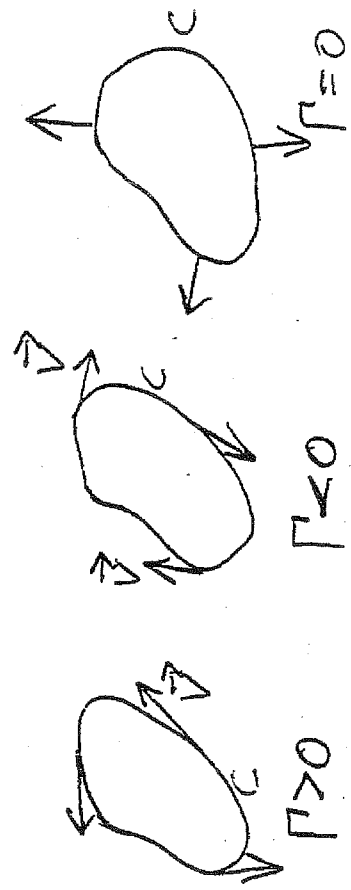
zirkulation:

$$\Gamma = \oint \vec{v} \cdot d\vec{r} = \oint v_e \parallel d\vec{r} \parallel$$

$\Gamma$ : Summe d. Tangentialkompon.  $v_t$  auf  $C$

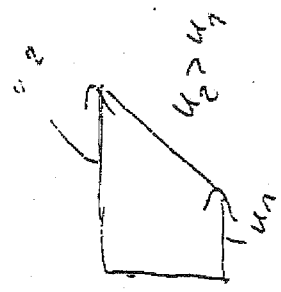
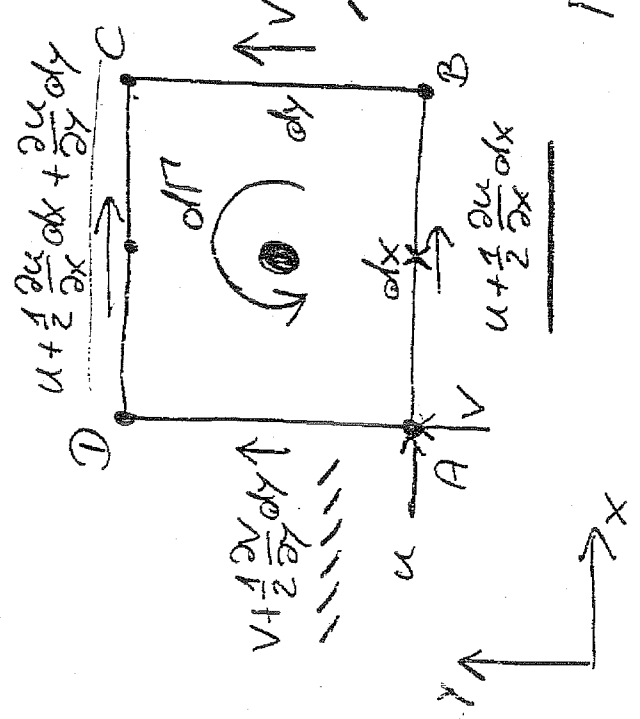


Amalure:  $C$  im Gegenuricht:  
 sin durchlaufen



Zusammenhang zw.

$\vec{\omega}$  und  $\Gamma$



$$v + \frac{1}{2} \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\Gamma = \oint \vec{v} \cdot d\vec{r}$$

$$d\Gamma = \left( u + \frac{1}{2} \frac{\partial u}{\partial x} dx \right) dy - \left( v + \frac{1}{2} \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) dx$$

$$+ \left( v + \frac{1}{2} \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) dx - \left( u + \frac{1}{2} \frac{\partial u}{\partial x} dx \right) dy$$

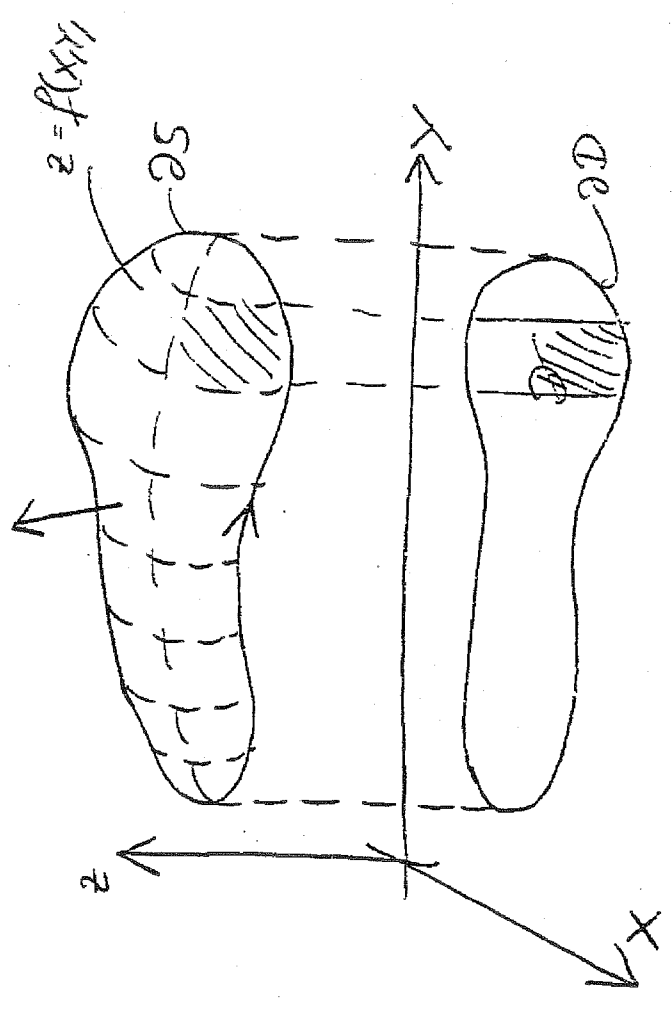
$$d\Gamma = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$d\Gamma = 2 \omega_z dx dy$$

$$\Rightarrow \Gamma = \oint_C \vec{v}_t \cdot d\vec{r} = 2 \int_A \omega_z dA$$

Zusammenhang Oberflächen:  
Integral und Linienintegral  
→ Stokesches Theorem

$$\int_{\partial S} \vec{\Phi}(\vec{r}) \cdot d\vec{r} = \int_S (\nabla \times \vec{\Phi}) \cdot \vec{n} dA = \int_S \text{rot } \vec{\Phi} \cdot \vec{n} dA$$



Es sei  $\vec{\omega} = \vec{\nabla} \times \vec{v} \Rightarrow \vec{\nabla} \times \vec{\omega} = \text{rot } \vec{v} = 2\vec{\omega}$   
 bzw.

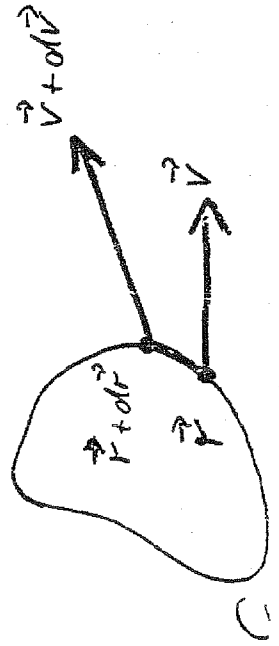
$$\Gamma = \oint_C \vec{v} \cdot d\vec{r} = \int_A \text{rot } \vec{v} \cdot \vec{n} \, dA$$

D.h.: Die Zirkulation entlang d. Randkurve einer beliebigen räumlichen Fläche ist gleich dem doppelten Wirbelfluß durch d. zugehörige Proj. fläche

$\frac{d\Gamma}{dt}$ : ? in reibungsfrei, barotroper Strömung.

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint_C \vec{v} \cdot d\vec{r} = \oint_C \frac{d\vec{v}}{dt} \cdot d\vec{r} + \oint_C \vec{v} \cdot \frac{d(d\vec{r})}{dt}$$

(im Komplex)  
 isentrop  $\rho p = \text{const}$



reibungsfrei Strömung. ( $\nu = 0$ )

$$\frac{d\vec{v}}{dt} = \vec{g} - \frac{1}{\rho} \vec{\nabla} p \quad \text{Euler Gln.}$$

$$\Rightarrow \oint_C \frac{d\vec{v}}{dt} \cdot d\vec{r} = \oint_C \vec{g} \cdot d\vec{r} - \oint_C \frac{1}{\rho} \vec{\nabla} p \cdot d\vec{r}$$

$$\left[ \vec{\nabla} p \cdot d\vec{r} = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = dp \right]$$

$$\oint_C \frac{d\vec{v}}{dt} \cdot d\vec{r} = \oint_C \vec{g} \cdot d\vec{r} - \oint_C \frac{dp}{\rho}$$

$$\frac{d\Gamma}{dt} = \underbrace{\oint_C \vec{g} \cdot d\vec{r}}_{\text{I}} - \underbrace{\oint_C \frac{dp}{\rho}}_{\text{II}} + \underbrace{\oint_C \vec{v} \cdot \frac{d(d\vec{r})}{dt}}_{\text{III}}$$

I: Satz: pot. Vektor-  
feld ist als Grad. zu schreiben  $\Rightarrow$   
 $\vec{g} = \vec{\nabla} f$

$$\oint_C \vec{g} \cdot d\vec{r} = \oint_C \vec{\nabla} f \cdot d\vec{r} = \oint_C df = 0$$

II: Annahme: Ström. ist barotrop

$$\rightarrow s = s(p)$$

$$s = s(p) = \frac{\partial f}{\partial p}$$

$$\text{Definition: } \frac{1}{s} \equiv \frac{\partial F}{\partial p} = \frac{\partial f}{\partial p}$$

$$\oint_C \frac{\partial F}{\partial p} dp = \oint_C dF = 0$$

III:  $d\vec{v} = \frac{d}{dt}(d\vec{r})$

$$\oint_C \vec{v} \cdot \frac{d}{dt}(d\vec{r}) = \oint_C \vec{v} \cdot d\vec{v} = \oint_C d\left(\frac{v^2}{2}\right) = 0$$

$$\frac{d\Gamma}{dt} = 0$$

reibg-frei, barotrope  
Ström. mit konst. Völ.

Satz v. Thomson

$$\Gamma(t=0) = 0 \rightarrow \Gamma(t+\Delta t) = 0$$

$\Rightarrow$  Ström. drehungsfrei (Stokes)

Achtg.: Gültig für d. Kurve C

Wirbeltransportgleichung

Für  $s = \text{konst}$ ,  $z = \text{konst}$

wird eine Glg. für  $\vec{\omega}$  abgeleitet.

Wirbelhaltungssatz

Identität:  $\text{div}(\text{rot } \vec{\Phi}) = 0$

Mit  $\vec{\omega} = \frac{1}{2} \text{rot } \vec{v}$

$$\text{div } \vec{\omega} = \vec{\nabla} \cdot \vec{\omega} = 0$$

Zusammenhang zwischen  $\vec{V}$  und  $\vec{\omega}$  führt auf

$\vec{\omega}$ -Gleichung aus

$$\text{rot}(\text{Impulserhaltung})$$

$$\text{Da } \text{rot}(\text{grad } h) = 0 \text{ gilt}$$

$$\text{rot}(\text{grad } f) = 0 \text{ Grav. f. f.}$$

$$\text{rot}(\text{grad } p) = 0 \text{ Druck f. f.}$$

$$\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{V}) + \vec{\nabla} \times [(\vec{V} \cdot \vec{\nabla}) \vec{V}] = \nu \nabla^2 (\vec{\nabla} \times \vec{V})$$

$$\vec{\nabla} \times \vec{V} = \vec{\omega}$$

Identität

$$\begin{aligned} (\vec{V} \cdot \vec{\nabla}) \vec{V} &= (\vec{\nabla} \times \vec{V}) \times \vec{V} + \frac{1}{2} \vec{\nabla} (\vec{V} \cdot \vec{V}) \\ &= \vec{\omega} \times \vec{V} + \frac{1}{2} \vec{\nabla} (u^2 + v^2 + w^2) \end{aligned}$$

$$\vec{\nabla} \times \left[ \frac{1}{2} \vec{\nabla} (u^2 + v^2 + w^2) \right] = 0$$

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{\omega} \times \vec{V}) = \nu \nabla^2 \vec{\omega}$$

$$\begin{aligned} \vec{\nabla} \times (\vec{\omega} \times \vec{V}) &= (\vec{V} \cdot \vec{\nabla}) \vec{\omega} - (\vec{\omega} \cdot \vec{\nabla}) \vec{V} + \\ &\quad \underbrace{\vec{\omega} \vec{\nabla} \cdot \vec{V}}_{=0} - \underbrace{\vec{V} \vec{\nabla} \cdot \vec{\omega}}_{=0} \\ &= (\vec{V} \cdot \vec{\nabla}) \vec{\omega} - (\vec{\omega} \cdot \vec{\nabla}) \vec{V} \end{aligned}$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{\omega} = \frac{\partial \vec{\omega}}{\partial t}$$

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} = \underbrace{(\vec{\omega} \cdot \vec{\nabla}) \vec{V}}_{(a)} + \nu \nabla^2 \vec{\omega}}_{(b)}$$

Wirbeltransportgleichung

(a): Ändg. d. Wirbelstärke durch Streckg. u. Neigung d. Wirbellinien

(b): Ändg. v.  $\vec{\omega}$  durch Diffusion

$$\vec{\omega} \perp \vec{v} \Rightarrow (\vec{\omega} \cdot \vec{\nabla}) \vec{v} = 0$$

$$\frac{d\omega}{dt} = \nu \nabla^2 \omega$$

Bsp.:  $\omega = \text{konst.}$

reibungsfree Ström.

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = 0$$

Im reibungslos Ström. (mit  $\nu = \text{konst.}$ ,  $\zeta = \text{konst.}$ ) wird Drehung weder erzeugt noch vernichtet.

Stationäre Ström.

$$u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = 0$$

Wirbeltransportgl. ist eine modifizierte Form d. Navier-Stokes Glgn.  $\Rightarrow \vec{\omega} = \vec{0}$  (obere freie Ström.) ist auch Lösung d. Impulserhaltung.

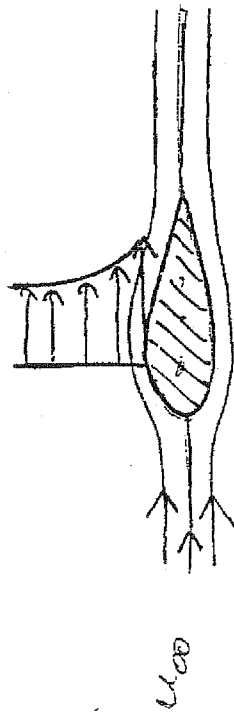
$\vec{\omega} = \vec{0} \Rightarrow$  Potentialströmung



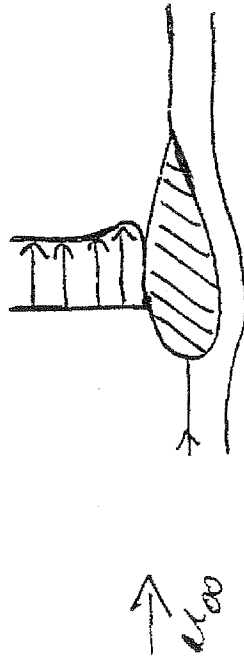
# Potentialströmungen

$s = \text{konst.}$ , reibungsfrei, 2D

$\rightarrow \frac{d\omega}{dt} = 0 \quad u_{\infty}$



idealisierte Strög.,  $\nu = 0$



realistische Strög.

Reibungseffekte nur in dünnen Wandablachten interessa

$\Rightarrow$  Aufteilung:

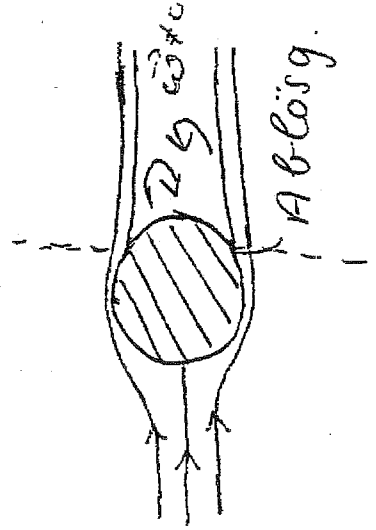
## • Außenbereich

- $\rightarrow$  Strög. wird reibungsfrei und drehungsfrei angenommen
- $\Rightarrow$  Analyse mittels d. Th. drehgr. Strög.

## • Innenbereich

- $\rightarrow$  Reibg. führt u. a. zur Diffusion d. Wirbelstärke

Weitere Strög.felder:



Ablösg.

Ablösg.

Potentialfunktion, Stromfunktion,  
Laplace Gleichung

$\vec{\omega} = \text{rot } \vec{v} = \vec{0}$  Lösung d. Impulsglg.

Identität:  $\text{rot}(\text{grad } f) = 0$

$\vec{v} = \text{grad } \Phi$  :  $u = \frac{\partial \Phi}{\partial x}, v = \frac{\partial \Phi}{\partial y}$

$\vec{\omega} = \text{rot}(\text{grad } \Phi) = 0$   
 $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$

$\text{div } \vec{v} = 0$  Bestpflg. v.  $\Phi$   $\rightarrow \text{div}(\text{grad } \Phi) = 0$

$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$   
Laplace Glg. d. Pot. fktn.

$u = \frac{\partial \Phi}{\partial x}, v = \frac{\partial \Phi}{\partial y}$

Definition von  $\psi$ :

$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$

• Konti. glg. erfüllt

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

• Bestpfl. v.  $\psi$  durch  $\vec{\omega} = 0$

$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$  (eben)

$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$   
Laplace Glg. d. Stromfktn.

Vgl. v.  $\phi$  und  $\psi$

$u = u, v = v$   
 $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

Cauchy-Riemannsche DGL

$\phi(x,y) = \text{konst.} \rightarrow$  Potentiallinien.  
 $\psi(x,y) = \text{konst.} \rightarrow$  Stromlinien

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy = 0 \quad (1)$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy = 0 \quad (2)$$

$$(1) \quad \frac{dy}{dx} \Big|_{\phi} = -\frac{u}{v} \quad (2) \quad \frac{dy}{dx} \Big|_{\psi} = \frac{v}{u}$$

$$\Rightarrow \frac{dy}{dx} \Big|_{\psi} = - \left[ \frac{dy}{dx} \Big|_{\phi} \right]^{-1} = \psi \perp \phi$$

(bzw. Cauchy-Riemann:  $\vec{\nabla} \phi \cdot \vec{\nabla} \psi = 0$ )

Stromlinien können sich nicht schneiden  $\rightarrow$  keine Komp. normal zu  $\psi = \text{konst.}$

Somit gilt



$$V_n = \frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial s} = 0$$

$$V_t = \frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial n} \neq 0$$

kinemat.  
Randbedg.

$\rightarrow$  Fluidelemente gleiten auf d. Kontur.

Berechnung d. Volumenstrom über  $\psi$



$$-V_n \rightarrow \frac{\partial \psi}{\partial s} ds$$

$$d\psi = \frac{\partial \psi}{\partial n} dn = V_t dn$$

$$d\dot{Q} = b d\psi$$

$$\uparrow \quad d\dot{Q} = b V_t dn$$

## Berechnung d. Druckverteilung

$\phi$  od.  $\psi$  bekannt  $\Rightarrow u, v$

Impulserhaltung für reibglos freie Ström. (Euler Gln.)

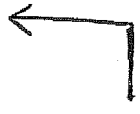
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = \vec{b} - \frac{1}{\rho} \nabla p$$

Identität:

$$(\vec{v} \cdot \nabla) \vec{v} = \nabla \frac{v^2}{2} - \vec{v} \times (\nabla \times \vec{v})$$

drehungsfrei:

$$\vec{\omega} = \nabla \times \vec{v} = \vec{0}$$



Potential:

$$\vec{b} = -\nabla(\varphi z)$$

stationär:

$$\frac{\partial}{\partial t} \rightarrow 0$$

einsetzen und integrieren

$$p + \frac{\rho}{2} v^2 + \rho g z = \text{konst.} \\ (\text{Bernoulli})$$

$\Rightarrow$  Gln. sind linear  $\rightarrow$  Superposition möglich

Elementarström. mit  $\phi_1, \dots, \phi_n$  werden zu neuen Ström. zusammengesetzt.

$$\phi = \sum_{i=1}^n a_i \phi_i$$

$\phi_1, \phi_2, \dots, \phi_n$  bekannt,  $a_i$  erp. gepaßt am Problem  $\Rightarrow \phi$

## Komplexe Potentialfunktion

kurz: Theorie der komplexen Funktionen

Wichtig für d. Analyse der Laplaceschen Differentialgl.

Definition:

Kompl. Geschw.:  $w = u + iv$

Konjug. kompl. Geschw.:  $\bar{w} = u - iv$

Die Funktion

$$F(z) = \int \bar{w} dz$$

ist ol. komplexe Potentialfunktion;  
 $F(z)$  erfüllt ol. Laplace Glg.

$$dz = dx + i dy, \quad u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}$$

$$F(z) = \int (u dx + v dy) + i \int (u dy - v dx)$$

$$\underline{\underline{F(z)}} = \int d\phi + i \int d\psi = \phi(x,y) + i\psi(x,y)$$

Laplace Glg.: ?

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial F}{\partial z}, \quad \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial z^2}$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = \frac{\partial F}{\partial z} i, \quad \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial z^2} i^2$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial z^2} = 0!$$

bzw.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + i \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = 0$$

$\Rightarrow F(z)$  beschreibt Potentialström.

$F(z)$ : analytische Funktion

Singulartät:  $F(z)$  od.  $\frac{dF}{dz}$

$\rightarrow 0$  oder  $\rightarrow \infty$ ;

Cauchy-Riemann

nicht erfüllt

$\Rightarrow \phi = \text{konst u. } \psi = \text{konst}$

nicht orthogonal

Bsp.:  $F(z) = \ln z$  analyt. außer in  $z=0$

•  $F(z) = \frac{1}{z}$  analyt. außer in  $z=0$

Superpositionsprinzip auch für  $F(z)$ ;

$$F(z) \text{ bekannt} \Rightarrow \frac{dF}{dz} = \bar{w} = u - iv$$

$\Rightarrow$  Geschw. vertlg.  $\Rightarrow$  Druck vertlg.

Vorgehensweise (hier): Vorgabe v.  $F(z)$

Beispiele für die komplexe Potentialfunktion

2D, inkompressibel

Winkel- und Eckenströmung

$$\begin{aligned} F(z) &= \frac{\alpha}{n} z^n = \frac{\alpha}{n} (x+iy)^n = \phi + i\psi \\ &= \frac{\alpha}{n} (re^{i\varphi})^n = \frac{\alpha}{n} r^n [\cos(n\varphi) + i \sin(n\varphi)] \end{aligned}$$

$$n \in \mathbb{R}, \quad \alpha = \alpha_r + i\alpha_i$$

zunächst  $a \in \mathbb{R}$

$$\phi = \frac{\alpha}{n} r^n \cos(n\varphi)$$

$$\psi = \frac{\alpha}{n} r^n \sin(n\varphi)$$

Strömlinien:  $\psi = \text{konst.}$

$$\rightarrow r^n \sin(n\varphi) = \text{konst.} \neq 0$$

$$\Rightarrow \sin(n\varphi) = 0$$

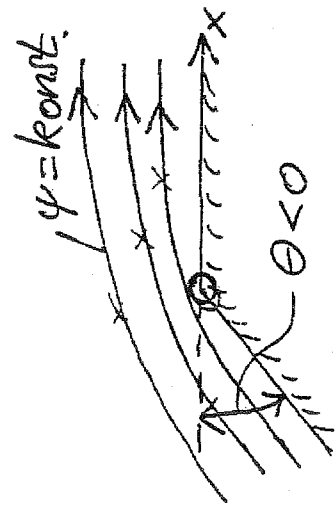
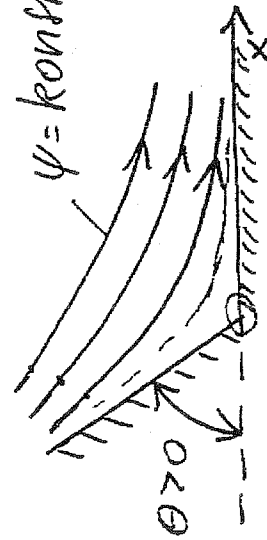
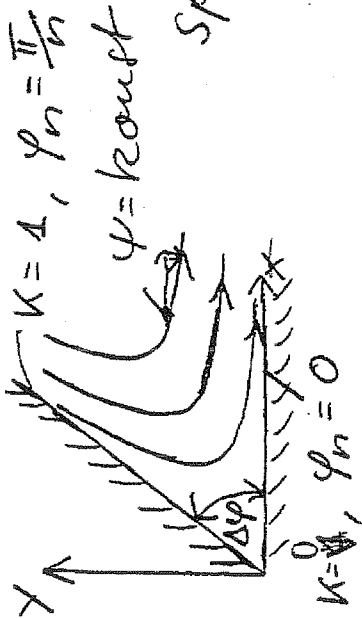
$$\varphi = \varphi_n = \kappa \left( \frac{\pi}{n} \right) \quad \kappa = 0, 1, \dots$$

$$\left. \begin{aligned} \kappa = 0 : \varphi = 0 \\ \kappa = 1 : \varphi = \frac{\pi}{n} \end{aligned} \right\} \Delta\varphi = \frac{\pi}{n}$$

$$\underline{n \geq 2} : \quad \Delta\varphi \leq \frac{\pi}{2}$$

$$\underline{2 > n > 1} : \quad \frac{\pi}{2} < \Delta\varphi < \pi$$

$$\underline{1 > n > \frac{1}{2}} : \quad \pi < \Delta\varphi < 2\pi$$



$$\frac{dF}{dz} = \bar{w}(z) = a z^{n-1}$$

bzw.  $= u - iv \quad z = r e^{i\varphi}$

$$\|\vec{v}\| = |a| r^{n-1}$$

konkave Ecke ( $n > 1$ ):  $\|\vec{v}\| \rightarrow 0$ ,  
sofern  $r \rightarrow 0$

konvexe Ecke ( $n < 1$ ):  $\|\vec{v}\| \rightarrow \infty$   
für  $r \rightarrow 0$

Strömlinien außerhalb von

$$\frac{dy}{dx} = \frac{v}{u} = -\tan[(n-1)\varphi]$$

# Parallelströmung $\pi$ $\Delta\varphi = \frac{\pi}{n}$

$$n = 4 \Rightarrow \Delta\varphi = \pi$$

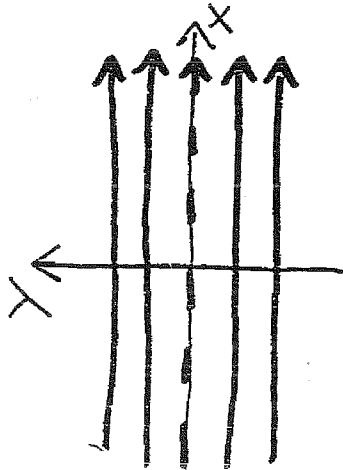
$$F(z) = \alpha z = (\alpha_r - i\alpha_i)(x + iy)$$

$$\frac{dF(z)}{dz} = \bar{w} = \alpha = \alpha_r - i\alpha_i = u - iv$$

$$\phi = \alpha_r x + \alpha_i y, \quad \psi = \alpha_r y - \alpha_i x$$

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = \alpha_r$$

$$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = \alpha_i$$

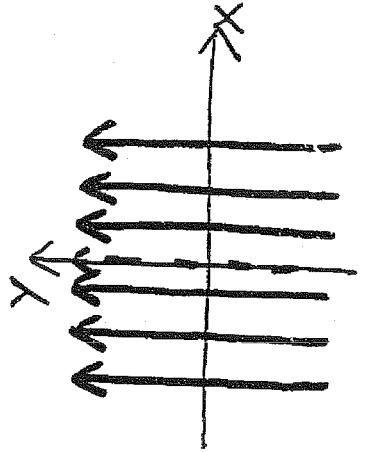


$$\alpha_r > 0$$

$$\alpha_i = 0$$

$$\alpha_r = 0$$

$$\alpha_i > 0$$





# Ebene Strompunktströmung

$n = 2$

$\Rightarrow \Delta \varphi = \frac{\pi}{2}$

$F(z) = \frac{\alpha}{2} z^2 = \phi + i\psi$

$\phi = \frac{\alpha}{2} (x^2 - y^2)$

$\psi = \alpha x y$

$a \in \mathbb{R}$

$\psi = \text{konst.} : y \sim \frac{1}{x}$

$\rightarrow xy = \frac{\text{konst.}}{\alpha}$

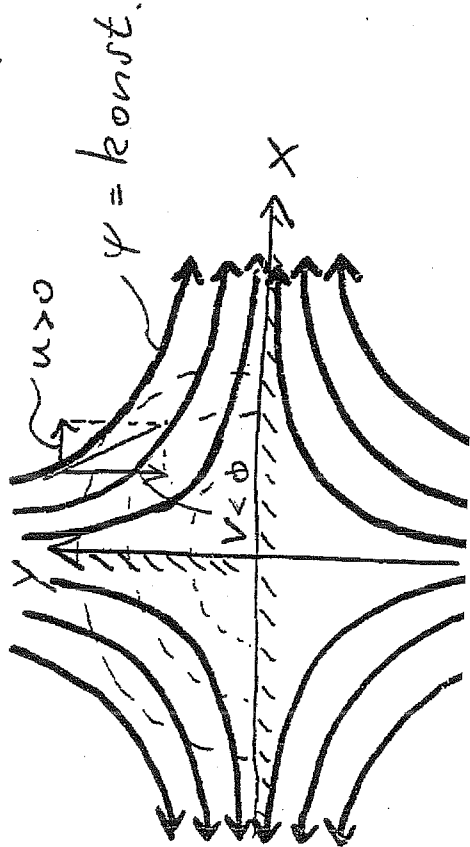
$xy = \text{const}$

$\rightarrow$  gleichseitige Hyperbel

$\bar{W}(z) = \alpha z = \alpha x + i\alpha y = u - iv$

$u = \alpha x$   
 $v = -\alpha y$

$\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x}$



Isotachen:  $\|\vec{v}\| = |\alpha| r$

Isobaren:  $p(r) = p_0 - \frac{\rho}{2} \alpha^2 r^2$

Quelle oder Senke

$F(z) = \alpha \ln z = \frac{E}{2\pi} \ln z = \frac{E}{2\pi} \ln(re^{i\varphi})$

$\Rightarrow \phi = \frac{E}{2\pi} \ln r$

$\psi = \frac{E}{2\pi} \varphi = \alpha \varphi$

$\phi = \text{konst.} : \text{Kreise um d. Ursprung}$

$\psi = \text{konst.} : \text{Strahlen durch d. Ursprung}$

$\bar{W}(z) = \frac{E}{2\pi z} = u - iv = \frac{E x}{2\pi z^2} - i \frac{E y}{2\pi z^2}$

$u = \frac{E x}{2\pi (x^2 + y^2)} \quad , \quad v = \frac{E y}{2\pi (x^2 + y^2)}$

gezeichnet in Polarkoordinaten

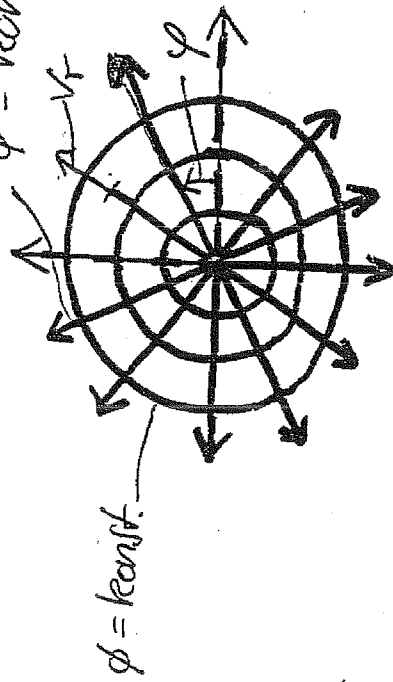
$$V_r = \frac{\partial \phi}{\partial r} = \frac{E}{2\pi r}$$

$$V_\phi = \frac{1}{r} \frac{\partial \phi}{\partial \phi} = 0$$

$E > 0$  : Quellströmung

$E < 0$  : Senkenströmung

$\psi = \text{konst.}$



$$\psi = a\phi$$

$$E = 2\pi a$$

$$a = \frac{E}{2\pi}$$

$$\dot{Q} = b [\psi(\phi + 2\pi) - \psi(\phi)]$$

$$= b [a(\phi + 2\pi) - a\phi] = 2\pi a b = \underline{\underline{Eb}}$$

$$\Rightarrow a = \frac{E}{2\pi} \quad \text{bzw.} \quad E = Q/b$$

$V_r \sim \frac{1}{r}$  :  $r \rightarrow 0 \Rightarrow V_r \rightarrow \infty$  (Singular.)

Potentialwirbel

Vertauschen von  $\phi$  und  $\psi$

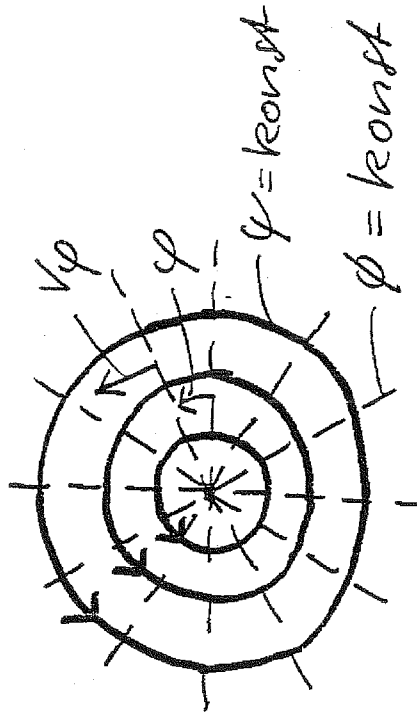
$$\Rightarrow \phi = c\psi$$

$$\psi = -c \ln r$$

$$F(z) = \phi + i\psi = -ic \ln z$$

$$V_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \phi} = 0$$

$$V_\phi = \frac{1}{r} \frac{\partial \phi}{\partial \phi} = -\frac{\partial \psi}{\partial r} = \frac{c}{r}$$



$c : ?$

rotational vortex

$$v_\varphi = \frac{c}{r} = \frac{\Gamma}{2\pi r}$$

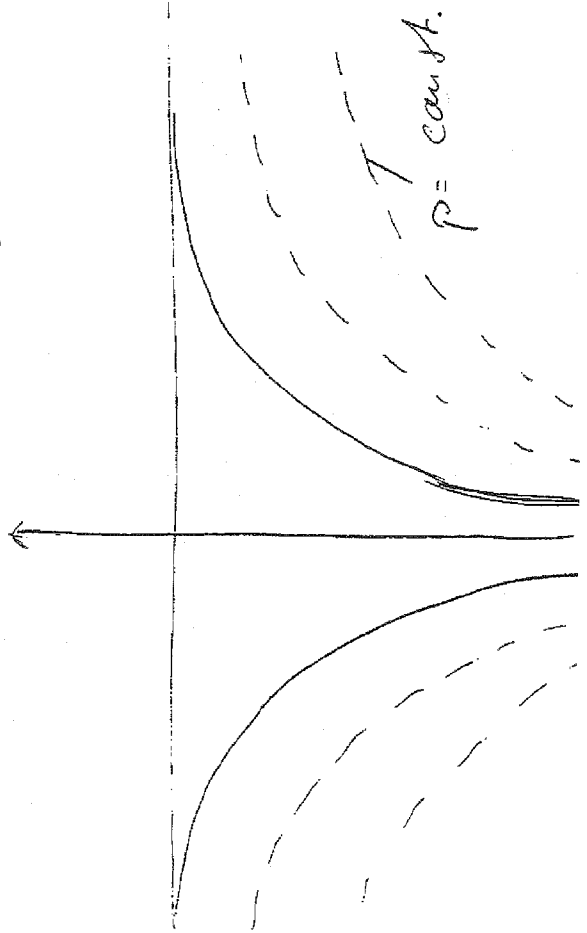
viscous stresses

$$\sigma_{r\varphi} = \mu \left[ r \frac{\partial v_r}{\partial \varphi} + r \frac{\partial}{\partial r} \left( \frac{v_\varphi}{r} \right) \right] = -\mu \frac{\Gamma}{\pi r^2} \neq 0$$

$\sigma_{r\theta} \neq 0$  since fluid elements do deform

however, the net viscous force vanishes since

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\mu (\nabla \times \vec{\omega})_i = 0 \quad \uparrow \quad \vec{\omega} = 0$$



(6)

(1)

$$\Gamma = \oint \vec{v} \cdot d\vec{r}$$

$$r = R: \quad \Gamma = \frac{c}{R} \cdot 2\pi R = 2\pi c$$

$$\Rightarrow F(z) = -i \frac{\Gamma}{2\pi} \ln z, \quad \phi = \frac{\Gamma}{2\pi} \varphi, \quad \psi = -\frac{c}{2\pi} \ln R$$

Wirbelkomponente in Polarkoordinaten

$$\omega_z = \frac{1}{2r} \left( \frac{\partial}{\partial r} (r v_\varphi) - \frac{\partial v_r}{\partial \varphi} \right)$$

$$\frac{\partial v_r}{\partial \varphi} = 0, \quad r v_\varphi = c = \frac{\Gamma}{2\pi} = \text{konst}$$

$$\Rightarrow \frac{\partial (r v_\varphi)}{\partial r} = 0 \quad [r \neq c]$$

$$\Rightarrow \omega_z = 0 \quad \text{für } r \neq 0$$

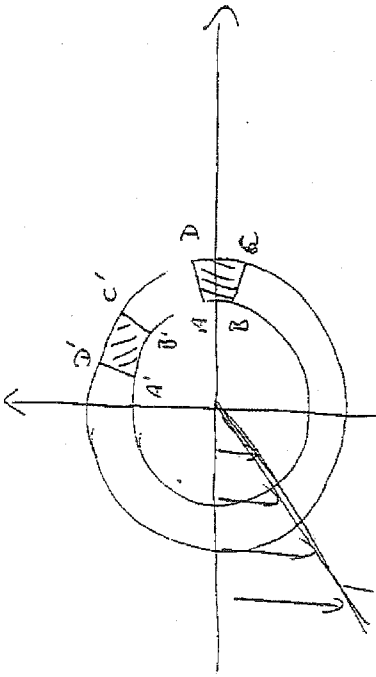
$$r = 0? \quad \omega_z \neq 0, \quad \text{denn } \Gamma \neq 0$$

$\Rightarrow$  Bei  $r=0$  ex. eine Wirbellinie oder Stabwirbel  $\perp$  zur Strömung

solid-body rotation

$$v_\varphi = \omega_0 r, \quad v_r = 0$$

$$\omega_z = \frac{1}{r} \frac{\partial}{\partial r} (r v_\varphi) = \frac{1}{r} \frac{\partial v_\varphi}{\partial r} = 2 \omega_0$$



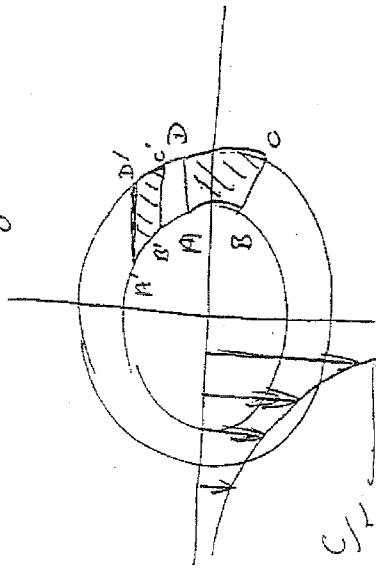
$$v_\varphi = \omega_0 r$$

$$\Gamma = \int \vec{v} \cdot d\vec{s} = \int_0^{2\pi} v_\varphi r \cdot r d\varphi = 2\pi r^2 v_\varphi = 2\pi r^2 \omega_0 r$$

irrotational vortex

$$v_\varphi = \frac{C}{r}, \quad v_r = 0$$

$$\Rightarrow \omega_z = \frac{0}{r}, \quad \Gamma = \int_0^{2\pi} v_\varphi r \cdot r d\varphi = 2\pi C$$



$$v_\varphi = \frac{C}{r}$$

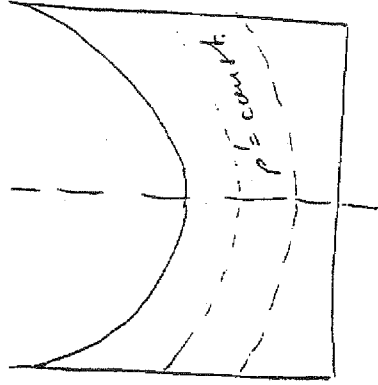
## role of viscosity

### solid-body rotation

fluid elements do not deform  
 $\Rightarrow$  viscous stresses are zero

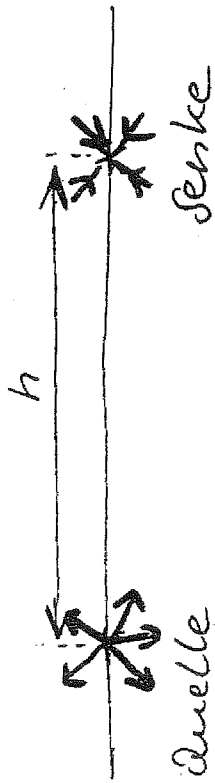
$$\sigma_{r\varphi} = \mu \left[ r \frac{\partial v_\varphi}{\partial r} + r \frac{\partial}{\partial r} \left( \frac{v_\varphi}{r} \right) \right] = 0$$

$\Rightarrow$  Euler eqs. are valid



Q (a)

# Dipolströmung

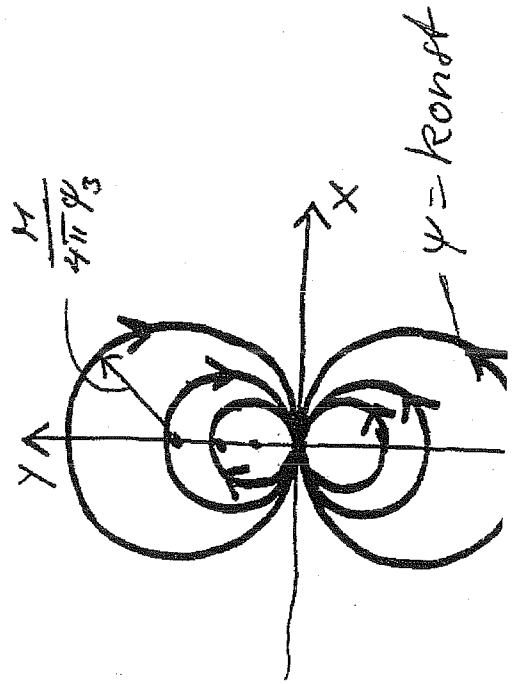


$E_{Quelle} = E_{Senke} = E$

Bedingung:  $E \sim \frac{1}{h}$  bzw.

$M = E h = \text{konst}$  für  $h \rightarrow 0$

$\Rightarrow$  Dipolströj. mit Dipolmoment  $M$  und Dipolachse  $x$ .



$F(z) = \frac{M}{2\pi} \frac{Eh}{z}$

$\lim_{h \rightarrow 0} \frac{u(z+h) - u(z)}{h}$

$\frac{d u(z)}{dz} = \frac{1}{z}$

$F(z) = \frac{M}{2\pi z} = \frac{M}{2\pi} \frac{x-iy}{r^2} = \phi + i\psi$

$\Rightarrow \phi = \frac{M}{2\pi} \frac{x}{x^2+y^2}, \quad \psi = -\frac{M}{2\pi} \frac{y}{x^2+y^2}$

$\psi$ -Gleichung liefert

$x^2 + \left(y + \frac{M}{4\pi\psi}\right)^2 = \left(\frac{M}{4\pi\psi}\right)^2$

$\Rightarrow$  Stromlinien: Kreise mit  
Zentren  $(0, -\frac{M}{4\pi\psi})$  u.  
Radien  $M/(4\pi\psi)$

(siehe Skizze)

$$e^{i\frac{\pi}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

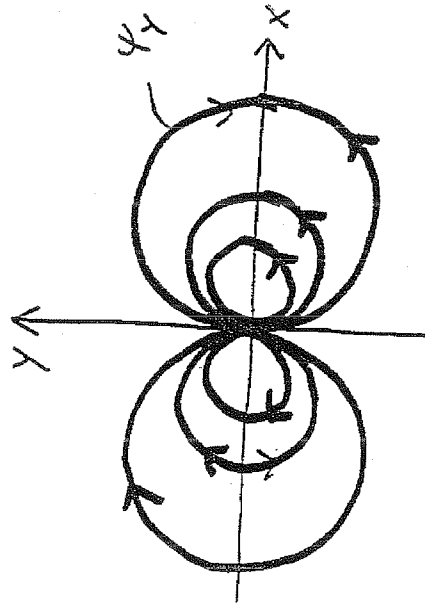
⇒ Multipl. mit  $i$  = Rotation um  $\frac{\pi}{2}$

D.h.  $iM$  statt  $M$  ⇒

$$\psi_y = \frac{Mx}{2\pi(x^2 + y^2)}$$

bzw.

$$\left(x - \frac{M}{4\pi\psi_y}\right)^2 + y^2 = \left(\frac{M}{4\pi\psi_y}\right)^2$$



Aus  $F(z) = \frac{M}{2\pi z}$  folgt

$$\bar{W}(z) = \frac{dF}{dz} = -\frac{M}{2\pi z^2} = u - iv$$

$$u = -\frac{M}{2\pi r^2} \cos(2\varphi) = -\frac{M}{2\pi} \frac{x^2 - y^2}{r^4}$$

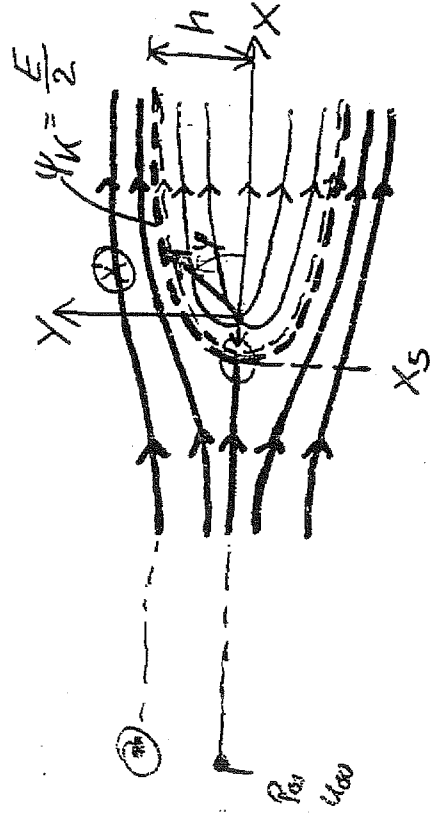
$$v = -\frac{M}{2\pi r^2} \sin(2\varphi) = -\frac{M}{2\pi} \frac{2xy}{r^4}$$

$\|\vec{v}\| \sim \frac{1}{r^2} \Rightarrow$  Singularität bei  $r=0$

Halbkörper

Parallelströmung + Quellenströg.

$$F(z) = u_{\infty} z + \frac{E}{2\pi} \ln z, \quad v_{\infty} = 0$$



$$\phi = u_{\infty} x + \frac{E}{2\pi} \ln r = u_{\infty} x + \frac{E}{4\pi} \ln(x^2 + y^2)$$

$$\psi = u_{\infty} y + \frac{E}{2\pi} \varphi$$

bzw. d. Geschw. komp.

$$u = \frac{\partial \phi}{\partial x} = u_{\infty} + \frac{E}{2\pi} \frac{x}{x^2 + y^2} \quad \text{⊗}$$

$$v = \frac{\partial \phi}{\partial y} = \frac{E}{2\pi} \frac{y}{x^2 + y^2}$$

⇒ Koordinaten des Hauptpunkts  
( $u = v = 0$ )

$$\begin{aligned} v=0 : \quad \psi_S = 0 & \quad \left| \begin{array}{l} \psi_S = 0, \text{ aus } \textcircled{*} \\ 0 = u_{\infty} + \frac{E}{2\pi} \frac{x}{x^2} \\ \Rightarrow \underline{\underline{\psi_S}} \end{array} \right. \\ u=0 : \quad \psi_S = -\frac{E}{2\pi} u_{\infty} & \end{aligned}$$

Wandstromlinie : ?

$\psi_K = \psi_S$  (Wrt von  $\varphi$  im Hauptpunkt)

$$\psi_S = \pi \Rightarrow \psi_S = \underline{\underline{\frac{E}{2}}}$$

$$\psi_K = u_{\infty} \overbrace{r \sin \varphi}^y + \frac{E}{2\pi} \varphi = \underline{\underline{\frac{E}{2}}}$$

$$\Rightarrow r_K = \frac{E}{2\pi u_{\infty}} \frac{\pi - \varphi}{\sin \varphi}$$

max. (Halb)breite  $h$  bei  $x \rightarrow \infty$ :

$$x \rightarrow \infty : \quad y = h, \quad \varphi = 0$$

$$\psi_{\infty} = \psi_K$$

$$\psi_{\infty} = u_{\infty} h = \psi_K = \left(\frac{E}{2}\right) \Rightarrow h = \frac{E}{2u_{\infty}}$$

Druckverteilung : ?

$$P_{\infty} + \frac{\rho}{2} u_{\infty}^2 = P + \frac{\rho}{2} (u^2 + v^2) \quad \text{bzw.}$$

$$c_P = \frac{P - P_{\infty}}{\frac{\rho}{2} u_{\infty}^2} = 1 - \left[ \left(\frac{u}{u_{\infty}}\right)^2 + \left(\frac{v}{u_{\infty}}\right)^2 \right]$$

$$C_p = 1 - \left[ \left( 1 + \frac{h}{\pi} \frac{x}{x^2 + y^2} \right)^2 + \left( \frac{h}{\pi} \frac{y}{x^2 + y^2} \right)^2 \right]$$

Druckverteilung auf d. Kontur:

$$u_k = u_{\infty} + \frac{E}{2\pi} \frac{c \cos \varphi}{r_k} = u_{\infty} + \frac{u_{\infty}}{\pi - \varphi} \sin \varphi \cos \varphi$$

$$v_k = \frac{E}{2\pi} \frac{\sin \varphi}{r_k} = \frac{u_{\infty}}{\pi - \varphi} \sin \varphi \sin \varphi$$

$$C_{p,k} = 1 - \left[ \left( 1 + \frac{c \cos \varphi \sin \varphi}{\pi - \varphi} \right)^2 + \left( \frac{\sin^2 \varphi}{\pi - \varphi} \right)^2 \right]$$

Additionstheoreme:

$$\sin(2\varphi) = 2 \cos \varphi \sin \varphi$$

$$\sin(2(\pi - \varphi)) = -\sin(2\varphi)$$

$$C_{p,k} = \frac{\sin(2\bar{\varphi})}{\bar{\varphi}} - \left( \frac{\sin \bar{\varphi}}{\bar{\varphi}} \right)^2$$

Staupunkt:  $\varphi = \pi$  bzw.  $\bar{\varphi} = 0$

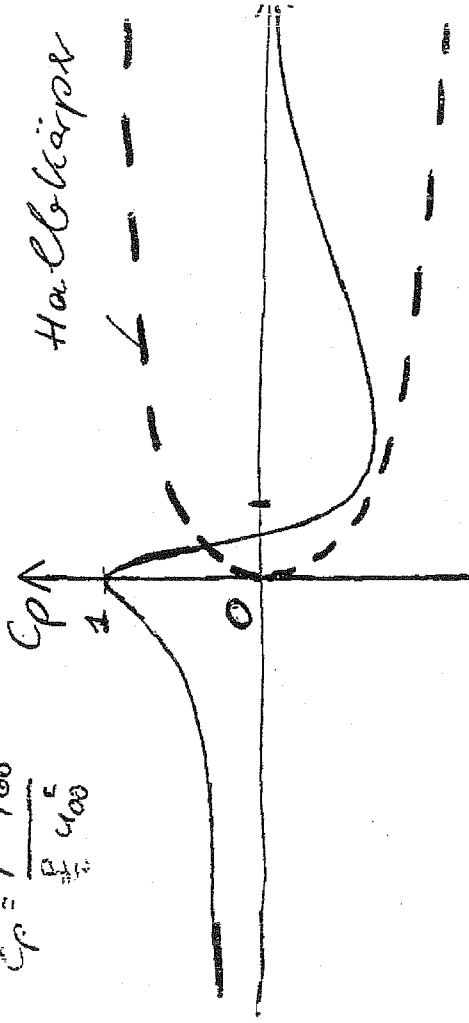
$$C_p = 1$$

$$C_p \left( \varphi = \frac{\pi}{2} \right) = 1 - \frac{4}{\pi^2}$$

$C_p \rightarrow 0$  für  $x \rightarrow \infty$

bzw.  $\varphi \rightarrow 0$

$$C_p = \frac{P - P_{\infty}}{\frac{\rho}{2} u_{\infty}^2}$$



Kreiszylinder

Parallelström + Quellen- + Senkenström  
 $E_a = E_s$

$\Rightarrow$  geschlossene Wandstromlinie



Abstand v. Quelle + Senke  $\rightarrow \infty$   
 $\Rightarrow$  Dipolström

$\rightarrow$  Parallelström + Dipolström.

$$F(z) = u_{\infty} z + \frac{M}{2\pi z}$$

$$\Rightarrow \phi = u_{\infty} x + \frac{M}{2\pi r^2} x$$

$$= \left( u_{\infty} + \frac{M}{2\pi r^2} \right) r \cos \varphi$$

$$\psi = u_{\infty} y - \frac{M}{2\pi r^2} y$$

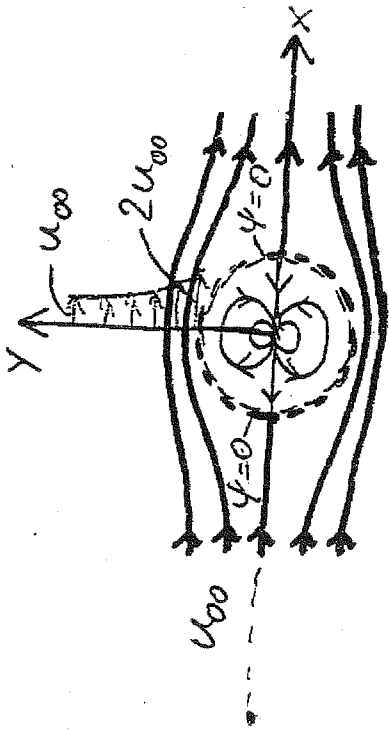
$$= \left( u_{\infty} - \frac{M}{2\pi r^2} \right) r \sin \varphi$$

$$\bar{w}(z) = \frac{dF}{dz} = u_{\infty} - \frac{M}{2\pi z^2}$$

bzw.

$$v_r = \frac{\partial \phi}{\partial r} = \left( u_{\infty} - \frac{M}{2\pi r^2} \right) \cos \varphi$$

$$v_{\varphi} = \frac{1}{r} \frac{\partial \psi}{\partial \varphi} = - \left( u_{\infty} + \frac{M}{2\pi r^2} \right) \sin \varphi$$



Staupunkte ( $v_r = v_{\varphi} = 0$ )

bei  $\varphi = 0$  und  $\varphi = \pi$

$$v_r = 0 \Rightarrow u_{\infty} - \frac{M}{2\pi R^2} = 0$$

$$R = \sqrt{\frac{M}{2\pi u_{\infty}}}$$

$\Rightarrow$  Wandstromlinie  $\psi = 0$

Geschw. auf  $\psi = 0$

$$v_r = 0$$

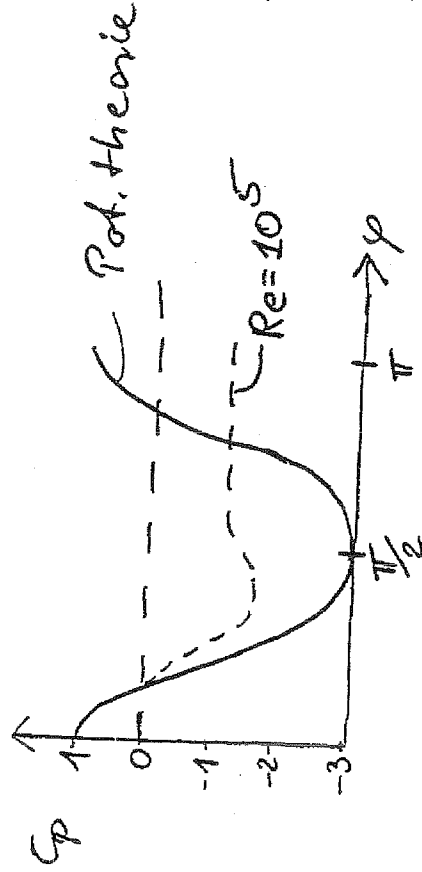
$$v_{\varphi} = -2 u_{\infty} \sin \varphi$$

$$\varphi = \frac{\pi}{2} \quad \text{od.} \quad \varphi = \frac{3}{2}\pi : |V_\varphi| = 2u_\infty$$

Druckverteilung auf Kontur:

$$P_k + \frac{\rho}{2} V_k^2 = P_\infty + \frac{\rho}{2} u_\infty^2$$

$$C_{P,k} = \frac{P_k - P_\infty}{\frac{\rho}{2} u_\infty^2} = 1 - \left(\frac{V_k}{u_\infty}\right)^2 = 1 - 4 \sin^2 \varphi$$



Pot.theorie  $\rightarrow$  symmetrischer  $p$ -Verteilung.

$\rightarrow$  keine resultierende Kraft

$\rightarrow$  d'Alambertsches Paradoxon

$$\Gamma < 4\pi u_\infty R \quad \Rightarrow \quad 2 \text{ Hauptpunkte}$$

$$\Gamma = 4\pi u_\infty R \quad \Rightarrow \quad 1 \text{ Hauptpunkt}$$

$$\Gamma > 4\pi u_\infty R \quad \Rightarrow \quad \text{keine Lösung auf d. Kontur, jedoch 1 freier Hauptpunkt}$$

$$V_r(r \neq R, \varphi) = 0 \quad \Rightarrow \quad \varphi = -\frac{\pi}{2}$$

$$V_\varphi(r \neq R, \varphi = -\frac{\pi}{2}) = 0 \quad \Rightarrow \quad u_\infty \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r} =$$

$$\Rightarrow r = \frac{1}{4\pi u_\infty} \left[ \Gamma \pm \sqrt{\Gamma^2 - (4\pi u_\infty R)^2} \right]$$

nur  $r > R$  berücksichtigen.

Addition eines Pot. wirbels  $\Rightarrow$  Seitenkraft

(12)

$$F(z) = u_{\infty} \left( z + \frac{R^2}{z} \right) + i \frac{\Gamma}{2\pi} \ln z$$

$$\Rightarrow \phi = u_{\infty} \cos \varphi \left( r + \frac{R^2}{r} \right) - \frac{\Gamma}{2\pi} \varphi$$

$$\psi = u_{\infty} \sin \varphi \left( r - \frac{R^2}{r} \right) + \frac{\Gamma}{2\pi} \ln r$$

$$V_r = \frac{\partial \phi}{\partial r} = u_{\infty} \cos \varphi \left( 1 - \frac{R^2}{r^2} \right)$$

$$V_{\varphi} = \frac{1}{r} \frac{\partial \phi}{\partial \varphi} = -u_{\infty} \sin \varphi \left( 1 + \frac{R^2}{r^2} \right) - \frac{\Gamma}{2\pi r}$$

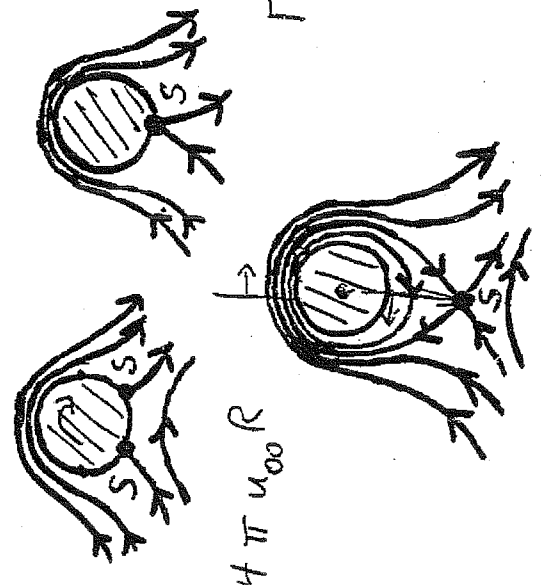
$$V_r = 0 \quad \text{für} \quad r = R$$

$$V_{\varphi}(r=R, \varphi) = -2 u_{\infty} \sin \varphi - \frac{\Gamma}{2\pi R}$$

Staupunkte: ?

$$V_{\varphi} = 0 \quad \rightarrow \quad \sin \varphi = - \frac{\Gamma}{4\pi u_{\infty} R}$$

(14)



$$\Gamma < 4\pi u_{\infty} R \quad \Gamma = 4\pi u_{\infty} R \quad \Gamma > 4\pi u_{\infty} R$$

Potentialwirbel  $\rightarrow \Gamma \rightarrow$  Seitenkraft

$\rightarrow$  Magnus Effekt

Druckverteilg. auf d. Kontur

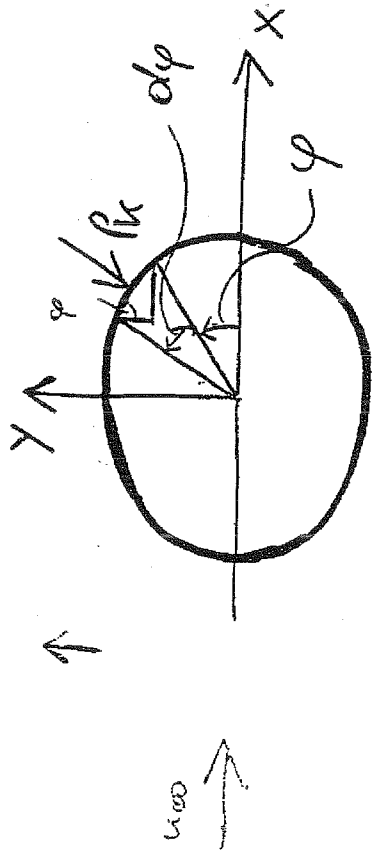
$$P + \frac{\rho}{2} (v_r^2 + v_{\varphi}^2) = P_{\infty} + \frac{\rho}{2} u_{\infty}^2$$

$$\Rightarrow P_{\text{K}} = P_{\infty} + \frac{\rho}{2} \left[ u_{\infty}^2 - \left( -2 u_{\infty} \sin \varphi - \frac{\Gamma}{2\pi R} \right)^2 \right]$$

bzw.

$$C_p = 1 - \left( 2 \sin \varphi + \frac{\Gamma}{2\pi u_{\infty} R} \right)^2$$

Kraft in  $y$ -Richtung:



$$L = - \int_0^{2\pi} p_k \sin \varphi R d\varphi$$

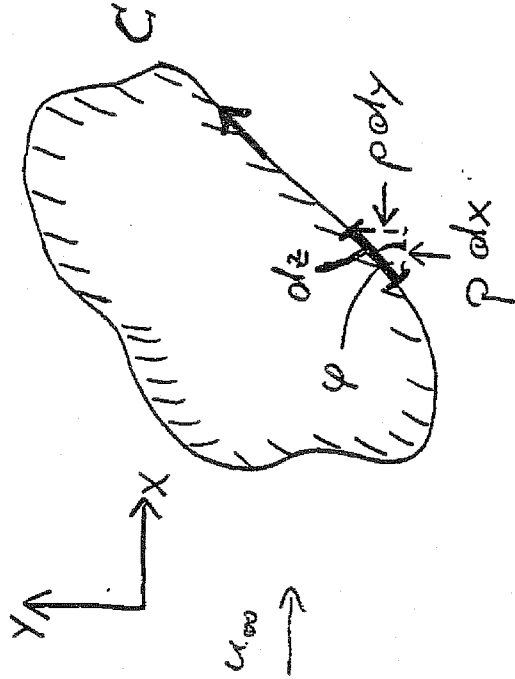
$$L = \frac{\rho u_{\infty} \Gamma}{\pi} \int_0^{2\pi} \sin^2 \varphi d\varphi = \frac{\rho u_{\infty} \Gamma}{2\pi} \int_0^{2\pi} (1 - \cos(2\varphi)) d\varphi$$

$$\boxed{L = \rho u_{\infty} \Gamma}$$

$L \sim \Gamma$  Kutta, Zhukowski

Auflösungssatz von  
Kutta - Zhukowski

Für beliebige ebene Körper  
gilt:  $L = \rho u_{\infty} \Gamma$



reibungsfree Strömung:

$$dD = -p dy$$

$$dL = p dx$$

infinitesimale Kraft

$$dD - i dL = p dy - i p dx = -i p d\bar{z}$$

$$d\bar{z} = dx - i dy$$

Gesamtkraft

$$D - iL = -i \oint_C p d\bar{z}$$

Druckverteilung

$$p_{00} + \frac{\rho}{2} u_{\infty}^2 = p + \frac{\rho}{2} (u^2 + v^2) = p + \frac{\rho}{2} (u+iv)(u-iv)$$

$$\Rightarrow D - iL = -i \oint_C \left[ p_{00} + \frac{\rho}{2} u_{\infty}^2 - \frac{\rho}{2} (u+iv)(u-iv) \right] d\bar{z}$$

$$\rightarrow \oint_C \left( p_{00} + \frac{\rho}{2} u_{\infty}^2 \right) d\bar{z} = 0$$

$$dz = dx + i dy = |dz| e^{i\varphi}$$

$$u+iv = \sqrt{u^2+v^2} e^{i\varphi}$$

$$\Rightarrow dz \parallel u+iv$$

$$\Rightarrow (u+iv) d\bar{z} \text{ reell und}$$

$$(u-iv) dz = (u+iv) d\bar{z}$$

$$\rightarrow D - iL = \frac{i}{2} \oint_C \phi(u-iv)^2 dz = \frac{i}{2} \oint_C \phi \left( \frac{dF}{dz} \right)^2 dz$$

(I. Blasius'sche Formel)  
gültig für eb., stat., drehsym. Ström.

Funktionentheorie:

Integr. auf jed. belieb. Kontur  
mögl., sofern keine Singularität  
zw. Körper und Wählkontur.

hier: • Ström. setzt sich u. a. aus  
Sing. zusammen, die jed.  
imhalb d. Körpers sich  
befinden.

- Wählkontur weit entfernt  
vom Körper

$$f(z) = u_{\infty} z + \frac{E}{2\pi} \ln z + \frac{i\Gamma}{2\pi} \ln z + \frac{M}{2\pi z} + \dots$$

$$\sum_i (Ea_i + E_{Si}) = 0 \quad \rightarrow \quad \text{Körperabfl. ist geschl.}$$

1. Blasiusche Formel:

$$D - iL = \frac{is}{2} \oint \left[ u_{\infty} + \frac{i\Gamma}{2\pi z} - \frac{M}{2\pi z^2} + \dots \right] dz$$

Residuensatz:

$$\oint_C f(z) dz = 2\pi i \cdot (\sum \text{Res}[f(z)])$$

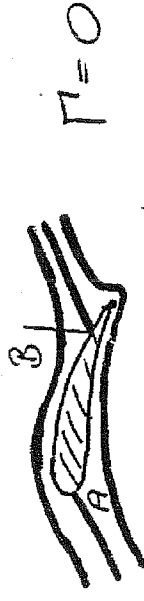
$\text{Res}[f(z)]$  : ?

$$\left[ u_{\infty} + \frac{i\Gamma}{2\pi z} \right]^2 = u_{\infty}^2 - \frac{\Gamma^2}{4\pi^2 z^2} + \frac{i u_{\infty} \Gamma}{\pi z}$$

$$\Rightarrow D - iL = \frac{is}{2} \left[ 2\pi i \left( \frac{i u_{\infty} \Gamma}{\pi} \right) \right]$$

$$\Rightarrow D = 0, \quad L = s u_{\infty} \Gamma$$

Entstehung der Zirkulation  
nur Körper mit scharfen Hinterran-  
ten erzeugen  $\Gamma$  (exp. Ergeb.)



Obige Darstellg.: ① drehpfr. Strög.,  $\Gamma$  steigt an

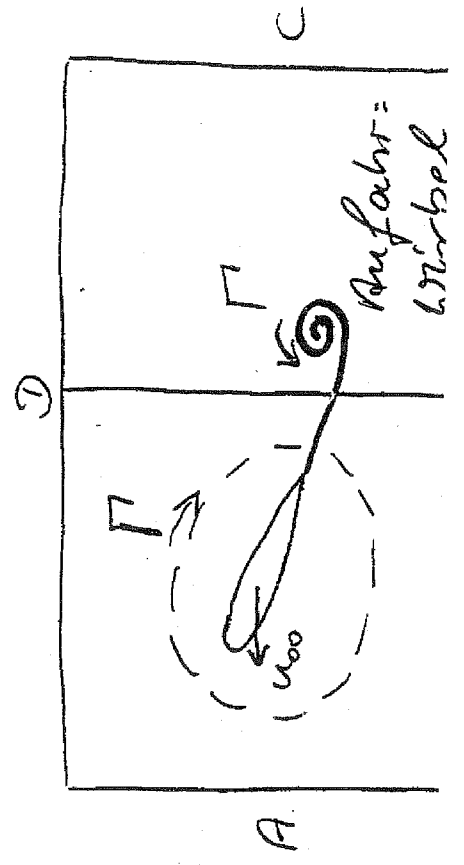
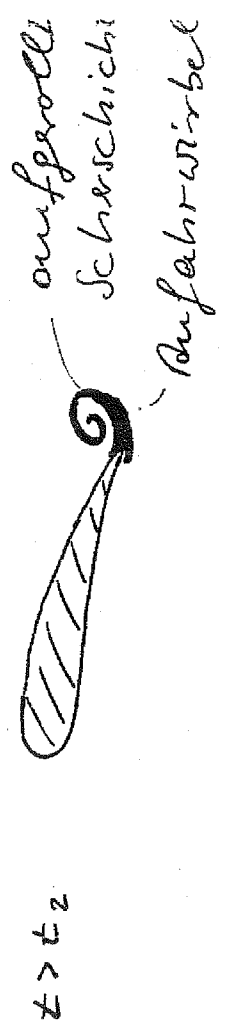
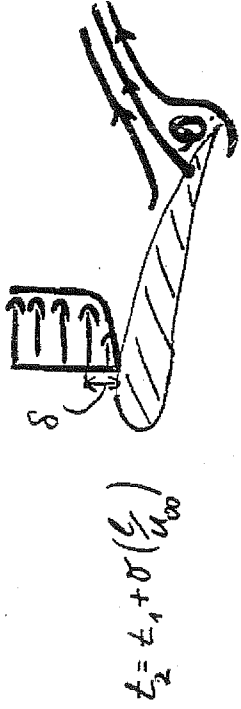
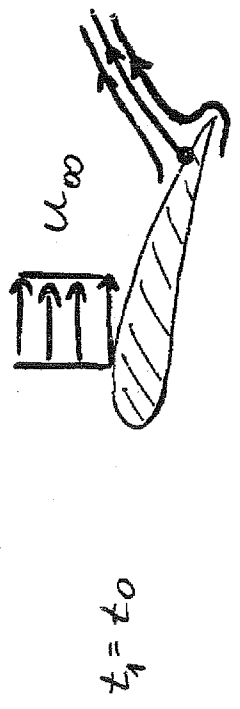
$\Gamma = 0$  Hauptkt. (B) auf d. Oberseite

$\Gamma \uparrow$  A, B wandern nach unten

$\Gamma_{\text{Kutter}}$  B auf d. Hinterecke  $\rightarrow$  glattes Abfließen d. Strög.

Hypothese von Kutta:  
 Strög. über eb., scharfkantigen Körper besitzt genau d.  $\Gamma$ , daß B auf d. Hinterecke liegt.  
 (Kutta Bedingung)

③ Warum gerührt eine realist. Strög. d. Kutta Bedingung?



Satz v. Thomson:  $\Gamma$  und jede geschl. Kurve ist konst., wenn d. Kurve in einer reibgr. Ström. liegt.

$\Rightarrow \Gamma_{ABCD}(t > t_0) = 0$ , da  $\Gamma_{ABCD}(t_0 = 0) = 0$

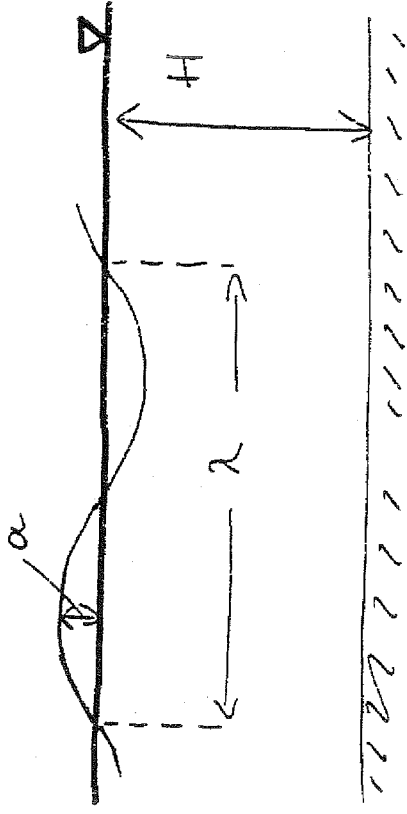
$\Rightarrow \Gamma_{ABD}$  komp.  $\Gamma_{BCD} = \Gamma_{\text{Anfahr}}$

nur  $\Gamma_{\text{Kutta}}$  bewirkt keine Oszilla-  
tion d. hint. Staupunkts.

D.h. Viskosität ruft zwar  $\mathcal{D}$   
hervor, jedoch ebenfalls  
 $\Gamma$  und somit  $L$ .

## Schwere Wellen

(4)



$H$ : mittlere Tiefe

•  $a/\lambda \ll 1$  und  $a/H \ll 1$

•  $H \gg \lambda$

•  $\gamma$  klein  $\rightarrow$  Reibg. kein Einfluß  
auf Wellenumbreitung.

• Beweg. aus d. Ruhe  $\rightarrow$  oberflächene  
Analyse



# Laminare Grenzschichten

analytische Lösen stat. Ström.

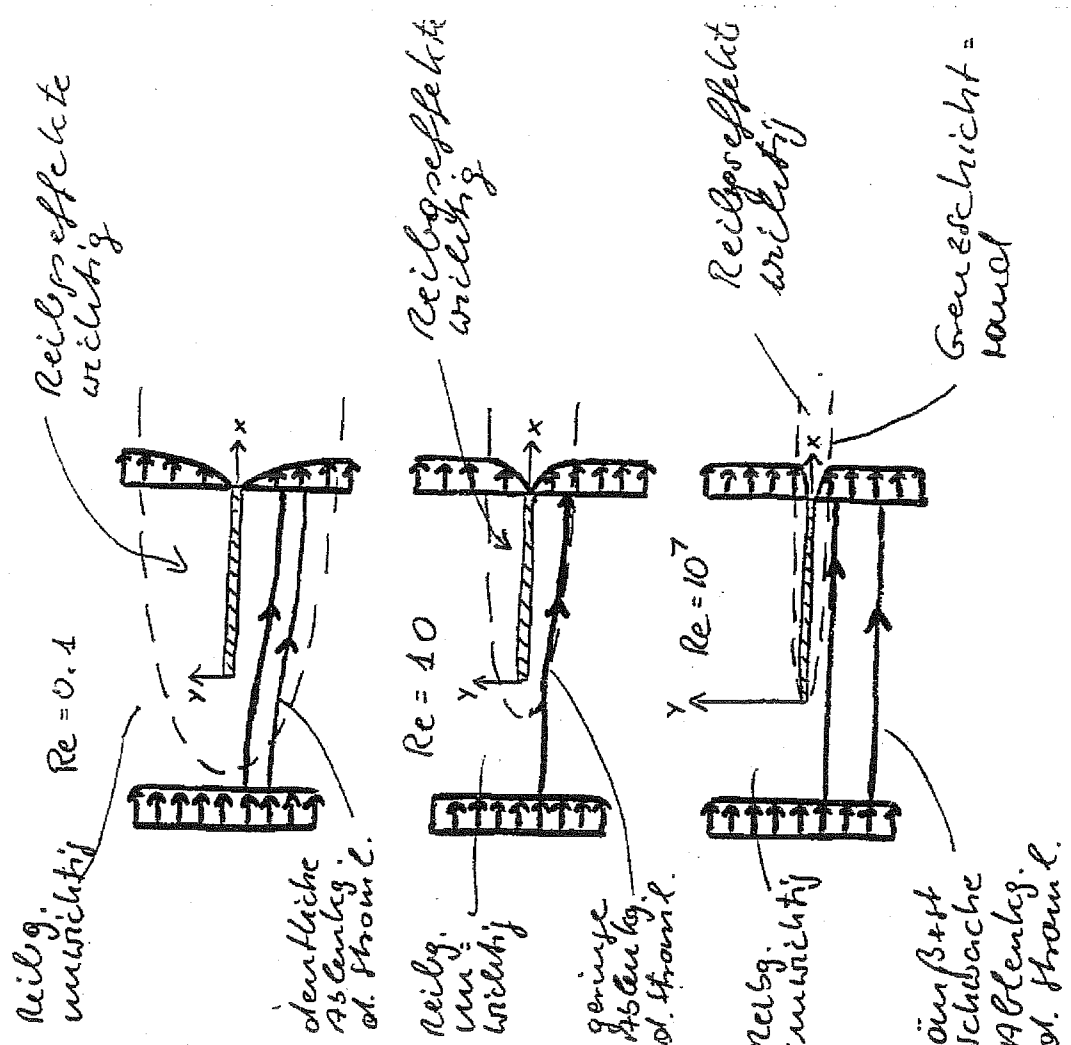
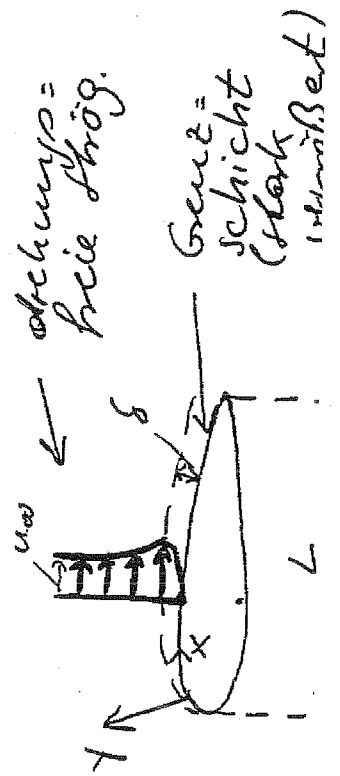
- Druck- u. Reibgestme im Gleichgewicht
- Trägheits- u. Druckkräfte im Gleichgewicht

1904 Grenzschichtkonzept v. Prandtl:

$\eta$  klein, dann werden Reibgestme Kräfte nur in unmittelbarer Wandnähe berücksichtigt.

$\eta \rightarrow 0$ , dann  $\delta \rightarrow 0$ ;

durch  $\delta$  wird d. Haftbedingung erfüllt  $\Rightarrow$  Widerstandskraft



# Grenzschichtgleichungen

klein bzw.  $Re$  groß

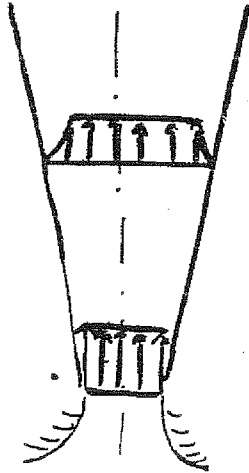
→ Grenzschicht existiert

$\delta \ll L \Rightarrow \frac{\partial u}{\partial x}$  sehr groß (i.a.)

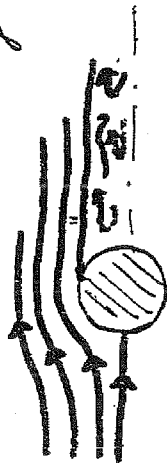
dünne Reibungsschichten:

- Wandgrenzschicht

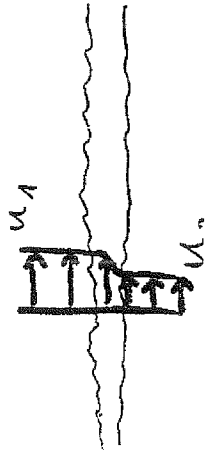
- Freistrah



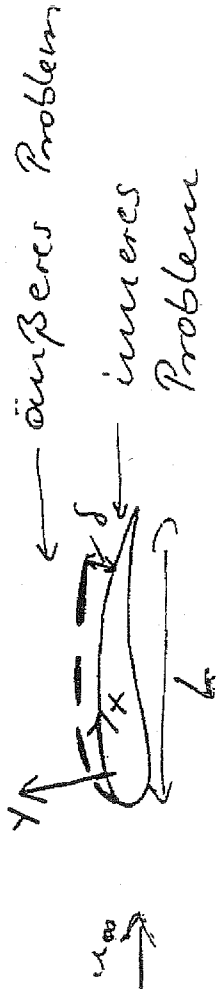
- Nachlauf



- Scherschicht



hier: Wandgrenzschichten



Ableitung eines Maßes für die Grenzschichtdicke  $\delta$

Imp. glg. in x-Richtung:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Referenzgeschw.  $u_{\infty}$   
charakt. Länge  $L$

$$\rightarrow u \frac{\partial u}{\partial x} \sim \frac{u_{\infty}^2}{L} \quad \text{Ordg. d. konvek. Term.}$$

Maß für Variation von  $v$  aus Kontinuitätsgl.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \gg v \quad \frac{\partial u}{\partial x} \ll \frac{\partial v}{\partial y}$$

$\Rightarrow$  von gl. Ordg. in der Grenzschicht

$$\Rightarrow \frac{u_{\infty}}{L} \sim \frac{v}{\delta} \Rightarrow v \sim \frac{u_{\infty} \delta}{L}$$

$$\Rightarrow v \frac{\partial u}{\partial y} \sim \frac{u_{\infty}^2}{L}$$

Ordg. d. konvektiven Terme  $\frac{u_{\infty}^2}{L}$

$$\text{Ordg. d. Reibgterme } \nu \frac{\partial^2 u}{\partial y^2} \sim \nu \frac{u_{\infty}}{\delta^2}$$

Annahme: Übereinstimmung d. Ordg.

$$\Rightarrow \delta \sim \sqrt{\frac{\nu L}{u_{\infty}}}$$

Vereinfachgn. d. Bewegungsglg. in

d. Grenzschicht

$$\text{es ist: } \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}, \quad \frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$$

Referenzgrößen:

Längen  $L$  in  $x$ ,  $\delta \ll L$  in  $y$ -Richt.

Druck  $\rho u_{\infty}^2$

Geschw.  $u_{\infty}$  in  $x$ ,  $\frac{u_{\infty} \delta}{L}$  in  $y$ -Richt.

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{\delta}$$

$$\bar{u} = \frac{u}{u_{\infty}}, \quad \bar{v} = \frac{\nu u_{\infty} / L}{\delta u_{\infty} / L} = \frac{\nu}{u_{\infty}} \sqrt{Re_L}$$

$$\bar{p} = \frac{p}{\rho u_{\infty}^2}$$

man erhält

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = - \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\frac{1}{Re} \left( \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) = - \frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{Re^2} \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{1}{Re} \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

wobei  $Re = u_{\infty} l / \nu$ .

dimensionslose Var. und die

Differentiale von Ordg. 1

$\Rightarrow Re \rightarrow \infty$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$0 = -\frac{\partial \bar{p}}{\partial \bar{y}}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

bzw. in dimensionsbehaft. Form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial y} = 0 \Rightarrow p_{\text{wand}} = p_f$$

Druckverteilung aus Euler Gg.

$$u \frac{du}{dx} = -\frac{1}{\rho} \frac{dp}{dx}$$

oder Bernoulli Gg.

$$p + \frac{\rho}{2} u^2 = \text{konst.}$$

$u$ : Geschw. bei  $\delta$ .

Aufangs- u. Randbedingungen:

$$u(x, 0) = 0, \quad v(x, 0) = 0$$

$$u(x, \infty) = u(x), \quad u(x_0, y) = u_0(y)$$

2D stationäre kompressible Ström.

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( 2 \frac{\partial u}{\partial y} \right)$$

$$\int_C p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( 2 \frac{\partial T}{\partial y} \right) + 2 \left( \frac{\partial u}{\partial y} \right)^2$$

bekannt:  $\rho(T)$ ,  $\lambda(T)$ ,  $\lambda(T)$ ,  $\frac{\partial \rho}{\partial x}$

Auflaufs- und Randbedingungen:

$$u(x, 0) = v(x, 0) = 0$$

$$T(x, 0) = T_w(x) \text{ oder}$$

$$\frac{\partial T}{\partial y} \Big|_{y=0} = 0$$

$$u(x, \infty) = u(x), \quad T(x, \infty) = T_i(x)$$

$$u(x_0, y) = u_0(y)$$

$$T(x_0, y) = T_0(y)$$

Gültigkeit d. Grenzschichtvereinfachg.

$$Re \gg 1$$

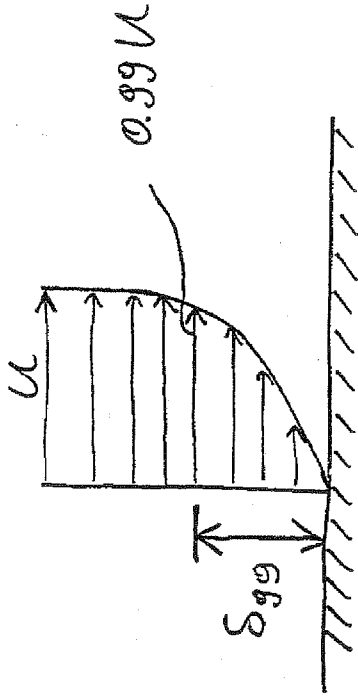
$$R \gg \delta$$

Grenzschichtgrößen

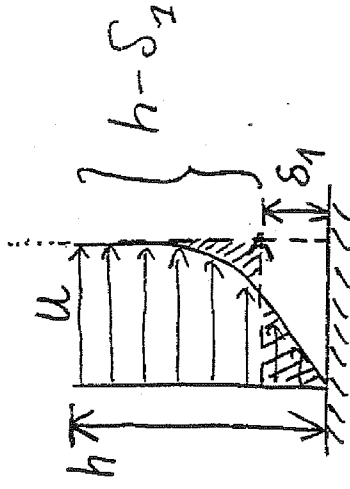
Definition d. Grenzschichtdicke  
(3 übl. Maße)

$0.99u$  - Dicke

$\delta \doteq \delta_{99}$ , wobei  $u(x, y = \delta_{99}) = 0.99u$



Verdrängungsdicke  $\delta_1$



Aufdünnung einer Außenströmung  
reißfr. Strömung um  $\delta_1$ , wobei  
 $\dot{Q}_i = \dot{Q}_v$

$h \gg \delta$ , es gilt

$$\int_0^h u \, dy = u(h - \delta_1)$$

$\dot{Q}_v$

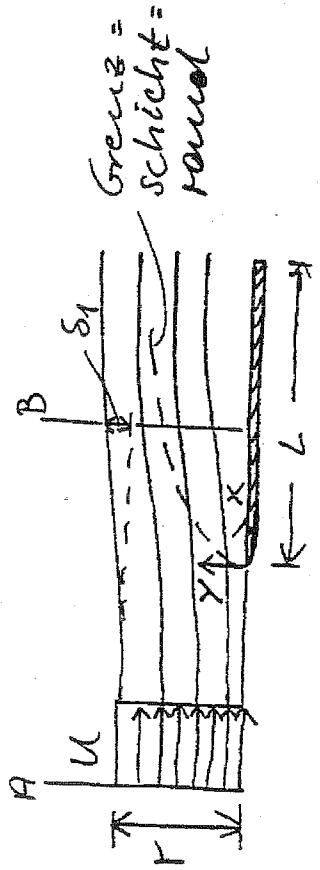
$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

$h \rightarrow \infty \Rightarrow$

für  $\frac{dp}{dx}$  ist  $\delta_1$  unterschiedl.,

weitere Interpretation von  $\delta_1$ :

$\delta_1$  ist d. Abstand, um den d. Ström. bei  $y > \delta$  abgedrängt warden.



$\dot{Q}$  in A und B:

$$U r = \int_0^{r+\delta_1} u \, dy = \int_0^r u \, dy + \int_0^{\delta_1} u \, dy$$

$$\Rightarrow U \delta_1 = \int_0^r (U - u) \, dy$$

$r \rightarrow \infty$ :

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Impulswertstärke  $\delta_2$

$\delta_2$  ist d. Dicke, so daß  $\int_0^{\delta_2} \rho u^2 \, dy$  d. Imp. verlust durch d. Grenzschicht darstell.

mit ob. Abb.:  $I_{v,b} \equiv$  Imp. fluß / Einheitsbr.

Schnitt A:  $I_{v,b,A} = \int_{r+\delta_1}^r \rho u^2 \, dy$

Schnitt B:  $I_{v,b,B} = \int_0^r \rho u^2 \, dy = \int_0^r \rho u^2 \, dy + \int_0^{\delta_1} \rho u^2 \, dy$

$\delta_2$  aus:  $I_{v,b,A} = I_{v,b,B}$

$$\int_0^r u^2 S_2 = \int_0^r u^2 r - \int_0^r u^2 dy - \int_0^r u^2 \delta_1$$

$$u^2 S_2 = \int_0^r (u^2 - u^2) dy - u^2 \int_0^r (1 - \frac{u}{u}) dy$$

$$S_2 = \int_0^{\infty} u (1 - \frac{u}{u}) dy$$

$$\Rightarrow r \rightarrow \infty$$

$$S_2 < \delta_1$$

Von Kármánsche Integralbeziehung

exakte Lösung d. Grenzschichtsgl. selten

Näherungsverfahren auf d. Basis eines Integrals (v. Kármán 1921)

gesucht

$$\int_0^h (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) dy = \int_0^h (u \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial y^2}) dy$$

mit  $h > \delta$

Umformung d. x- Imp. pgl. mittels  $u \frac{\partial u}{\partial x}$  und  $v \frac{\partial u}{\partial y}$

$$\underbrace{(u-u) \frac{\partial u}{\partial x}}_I + u \underbrace{\frac{\partial(u-u)}{\partial x}}_{II} + v \underbrace{\frac{\partial(u-u)}{\partial y}}_{III} = -\nu \underbrace{\frac{\partial^2 u}{\partial y^2}}_{IV}$$

$$\text{I: } \int_0^h (u-u) \frac{\partial u}{\partial x} dy = u \frac{\partial u}{\partial x} \int_0^h (1 - \frac{u}{u}) dy = u \delta_1 \frac{\partial u}{\partial x}$$

$$\text{III: } \int_0^h v \frac{\partial(u-u)}{\partial y} dy = v(u-u) \Big|_0^h - \int_0^h (u-u) \frac{\partial v}{\partial y} dy$$

$$= \int_0^h \frac{\partial u}{\partial x} (u-u) dy \quad \text{III'}$$

IV: Schubschl. auf d. Wand

$$v_0 = 2 \frac{\partial u}{\partial y} \Big|_{y=0}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$- \frac{2}{\delta} \int_0^h \frac{\partial^2 u}{\partial y^2} dy = - \frac{1}{\delta} \int_0^h \frac{\partial}{\partial y} \left( 2 \frac{\partial u}{\partial y} \right) dy = \frac{\tau_0}{\delta}$$

II, III':

$$\frac{\partial}{\partial x} [u(u-u)] = u \frac{\partial(u-u)}{\partial x} + (u-u) \frac{\partial u}{\partial x}$$

$$\int_0^h \left[ u \frac{\partial(u-u)}{\partial x} + \frac{\partial u}{\partial x} (u-u) \right] dy =$$

$$\int_0^h \frac{\partial}{\partial x} [u(u-u)] dy = \frac{d}{dx} u^2 \int_0^h \frac{u}{u} \left( 1 - \frac{u}{u} \right) dy = \frac{d}{dx} (u^2 \delta_2)$$

Zusammenfassung:

von Kärnänsche Integralbereich.

$$\frac{d}{dx} (u^2 \delta_2) + \delta_1 u \frac{du}{dx} = \frac{\tau_0}{\delta}$$

bzw.

$$\frac{d\delta_2}{dx} + \frac{1}{u} \frac{du}{dx} (2\delta_2 + \delta_1) = \frac{\tau_0}{\delta u^2}$$

Annahme: Geschwindigkeitsprof.

$$\Rightarrow \delta = f(x) \quad \text{und} \quad \tau_0 = g(x)$$

Beispiel: laminar aufströmte ebene Platte

$$\frac{du}{dx} = 0$$

$$\frac{d}{dx} \int_0^{\delta} (u-u)u dy = \frac{\tau_0}{\delta} \Leftarrow$$

Geschwindigkeitsprofil:

$$\frac{u}{U} = a + b \frac{y}{\delta} + c \left( \frac{y}{\delta} \right)^2 + d \left( \frac{y}{\delta} \right)^3$$

Randbedgn:

$$y=0 \quad u=0, \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

$$y \rightarrow \delta \quad u=U, \quad \frac{\partial u}{\partial y} = 0 \quad (2)$$



aus ① folgt:  $a = c = 0$

aus ② folgt:  $b = 3/2$

$d = -1/2$

$$\frac{u}{U} = \frac{3}{2} \frac{y}{\delta(x)} - \frac{1}{2} \left( \frac{y}{\delta(x)} \right)^3$$

$$U^2 \int_0^\delta \left( 1 - \frac{u}{U} \right) dy = \frac{39}{280} U^2 \delta$$

$$\frac{\partial^2 u}{\partial y^2} \Big|_{y=0} = 0$$

$$\frac{\partial u}{\partial y} \Big|_{y=0} = \frac{3}{2} \nu \frac{U}{\delta}$$

=, v. Kármán Beziehung f. d. eb. Platte:  $\frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - 2U \frac{\partial^2 u}{\partial x^2}$

$$\frac{39}{280} \frac{d}{dx} (U^2 \delta) = \frac{39}{280} U^2 \frac{d\delta}{dx} = \frac{3}{2} \nu \frac{U}{\delta}$$

Integr. in x-Richtung mit  $\delta=0$  bei  $x=0$

$$\delta = 4.64 \sqrt{\nu x / U}$$

$$\delta \sim \sqrt{x}$$

Reibungsbeiwert  $C_f = \frac{\tau_0}{\frac{\rho}{2} U^2}$

$$C_f = \frac{3/2 U \nu / \delta}{\frac{1}{2} U^2} = \frac{0.646}{\sqrt{Re_x}} \quad C_{f, \text{lam}} \sim x^{-1/2}$$

Ähnliche Lösung d. Grenzschicht =  
Strömung über ebenen Platte  
(Blasius Lösung).

Vernachlässigung d. Viskosität

$\rightarrow u = \text{konst} \rightarrow \frac{du}{dx} = 0$

$\rightarrow$  kein Längenausmaß

Vorstellung: Lösung ist "ähnlich"

$$\frac{u}{U} = f(\bar{x})$$

wobei:

$$\bar{x} = \frac{y}{\delta(x)}$$

(Ansatz von Blasius)

Ausgangsgleichungen und Randbedg.

$$\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \gamma \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u=0, \quad \gamma = u$$

$$0 \leq x \leq L, \quad \gamma = 0$$

$$0 \leq x \leq L, \quad \frac{\gamma}{S} \rightarrow \infty$$

contingf. via Stromfkt.  $\psi$

$$\frac{\partial \psi}{\partial x} = v, \quad \frac{\partial \psi}{\partial y} = -u$$

$$\psi = \int_0^y u dx = \int_0^y \int_0^L u dx dy = \int_0^L \int_0^y u dx dy$$

$$\psi = S u f(\bar{z}), \quad \infty \text{ auf } \bar{z} = \frac{\partial \psi}{\partial y}$$

Stromfunktion  $\psi$  in Imp.fg. einführen.

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = \gamma \frac{\partial^3 \psi}{\partial y^3}$$

$$x=0 \quad \frac{\partial \psi}{\partial x} = u$$

$$y=0 \quad \frac{\partial \psi}{\partial y} = \psi = 0$$

$$\frac{\gamma}{S} \rightarrow \infty \quad \frac{\partial \psi}{\partial x} = u$$

Rückführung auf  $f(\bar{z}) = \frac{\gamma}{S} u$

$$\frac{\partial^2 \psi}{\partial x^2} = u \left[ \frac{\partial^2 f}{\partial x^2} + S \frac{\partial^2 f}{\partial y^2} \right] = u \frac{\partial^2 [f - f']}{\partial x^2}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = u \frac{\partial^2 f}{\partial x \partial y} = -u \frac{\partial^2 [f - f']}{\partial x \partial y}$$

$$\frac{\partial \psi}{\partial y} = u f', \quad \frac{\partial^2 \psi}{\partial y^2} = u f'' = \frac{\partial^2 f}{\partial y^2}$$

mit  $\frac{\partial f}{\partial x} = \frac{\partial \bar{z}}{\partial x} = -f' \frac{\partial \bar{z}}{\partial x} = -\frac{\gamma}{S} \frac{\partial f}{\partial x}$   
 $\frac{\partial f}{\partial y} = \frac{\partial \bar{z}}{\partial y} = \frac{\partial f'}{\partial y} = f' \frac{\partial \bar{z}}{\partial y} = f' \frac{1}{S}$   
 $\frac{\partial}{\partial x} (f \bar{z}) = \bar{z} \frac{\partial f}{\partial x} + f \frac{\partial \bar{z}}{\partial x} = -\frac{\gamma}{S} \bar{z} \frac{\partial f}{\partial x} - \frac{\gamma}{S} f \frac{\partial \bar{z}}{\partial x}$   
 $\bar{z} = \frac{\gamma}{S}$

... ..

$$-u^2 f'' \frac{\partial}{\partial x} - u^2 \frac{f''}{\delta} \frac{\partial \delta}{\partial x} + u^2 \frac{f''}{\delta} \frac{\partial \delta}{\partial x} [f - f'] = \nu \frac{u f'''}{\delta^2}$$

$$- \frac{u \delta}{\nu} \frac{\partial \delta}{\partial x} f f'' = f''''$$

da  $f(\bar{x})$  folgt

$$\frac{u \delta}{\nu} \frac{\partial \delta}{\partial x} = \text{konstant} = \frac{1}{2} \quad (\text{gewählt})$$

$$\Rightarrow \delta = \sqrt{\frac{\nu x}{u}}$$

Somit gilt

$$\boxed{\frac{1}{2} f f'' + f'''' = 0}$$

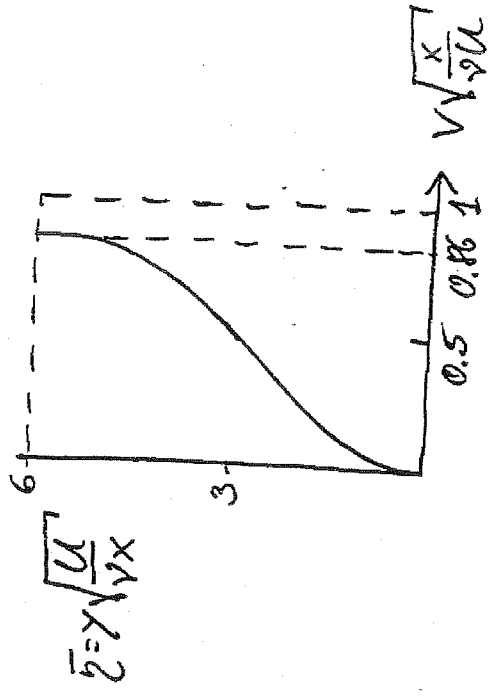
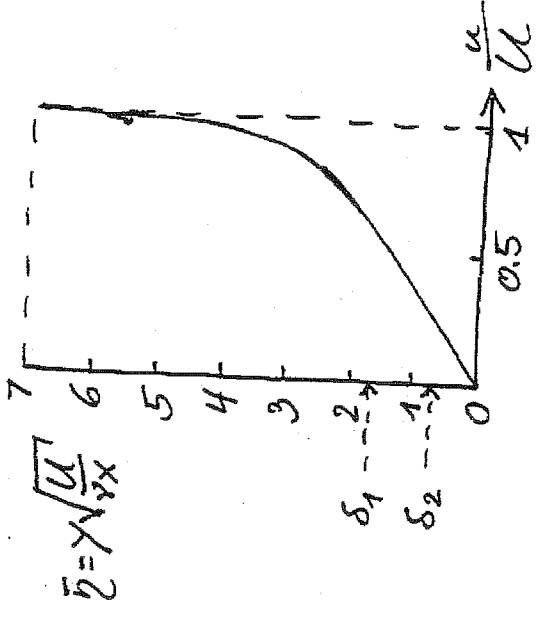
Randbedgn.

$$f(0) = f'(0) = 0$$

$$f'(\infty) = 1$$

Blasius 1908 über Reihenentwicklung gelöst.

$\frac{u}{U} = f'(\bar{x})$  : Geschw. prof. für sämtl. Schritte



$$v = -\frac{\partial \psi}{\partial x} = -u \frac{d\delta}{dx} [f - f']$$

$$= \frac{1}{2} \sqrt{\frac{\nu x}{u}} \left[ \frac{1}{2} \sqrt{\frac{\nu}{x u}} \right]$$

$$u = 0.99u \quad \text{bei } \bar{c} = 4.9$$

$$\Rightarrow S_{gg} = 4.9 \sqrt{\frac{\nu x}{u}} \quad \text{bzw.}$$

$$\frac{\delta_{gg}}{x} = \frac{4.9}{\sqrt{Re_x}} \Rightarrow \delta \sim \sqrt{x}$$

$$u = 1 \frac{m}{s} \rightarrow Re_{x=4m} = 6 \cdot 10^4 \Rightarrow S_{gg} = 2 \text{ cm}$$

$$\delta_1 = \int_0^{\infty} (1 - \frac{u}{u}) dy$$

$$\delta_1 = 1.72 \sqrt{\frac{\nu x}{u}}, \quad \delta_2 = 0.664 \sqrt{\frac{\nu x}{u}}$$

$$\tau_0 = 2 \frac{\partial u}{\partial y} \Big|_{y=0} = 2 \frac{\partial^2 \psi}{\partial y^2} \Big|_{y=0} = 2 \frac{u f''(0)}{\delta}$$

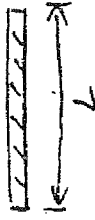
$$\tau_0 = \frac{0.332 \rho u^2}{\sqrt{Re_x}} \Rightarrow \tau_0 \sim \frac{1}{\sqrt{x}}$$

Achtg.:  $x \rightarrow 0 \quad \tau_0 \rightarrow \infty; \quad \frac{\partial^2 x}{\partial y^2}$

lokale Reibkoeffizient

$$c_f = \frac{\tau_0}{\frac{1}{2} \rho u^2} = \frac{0.664}{\sqrt{Re_x}}$$

Reibkraft auf einer Seite der Platte



$$D = \int_0^L \tau_0 dx = \frac{0.664 \rho u^2 L}{\sqrt{Re_L}}$$

$$D \sim u^{3/2}$$

Reibkoeffizient  $c_D$

$$c_D = \frac{D}{\frac{1}{2} \rho u^2 L} = \frac{1.33}{\sqrt{Re_L}}$$

wobei  $c_D = \frac{1}{L} \int_0^L c_f dx$

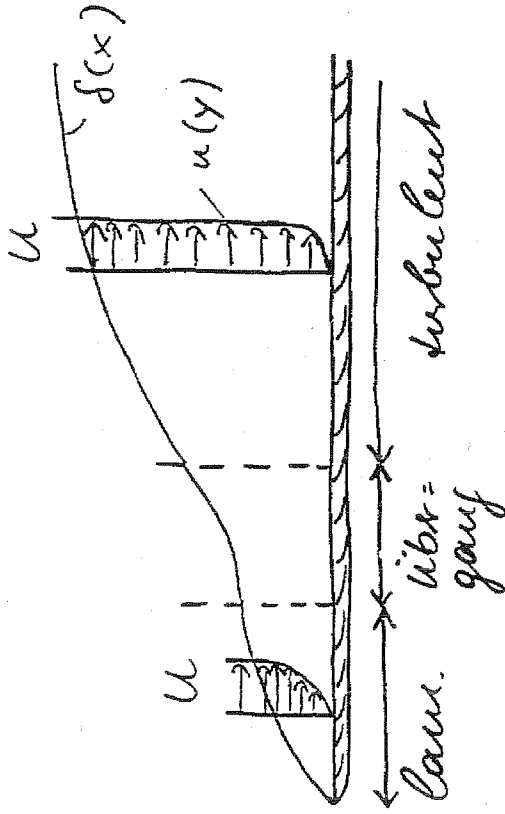
# Turbulente Grenzschichten

Vgl. Blasius - Exp. bis  $x \leq x_{kr}$

bzw.  $Re_x \leq Re_{kr}$  OK!

$Re_x > Re_{kr}$  Übergang lam.  $\rightarrow$  turb.

$Re_{x_{kr}} = f(\text{Oberfläche, Geom. od. Vorderkante, } u', v', \text{ etc.})$



längsgericht. ebene Platte:  $Re_{kr} \approx 5 \cdot 10^5$

lam. Grenzschicht:  $\delta \sim x^{1/2}$

$\tau_0 \sim u^{3/2}$

turb. Grenzschicht:  $\delta \sim x^{4/5}$

$\tau_0 \sim u^{7/4}$

Grenzschichtgleichungen  
der turbulenten Strömung

Erhaltungsgleichungen gültig  
 für lam. und turb. Strömung;

jedoch: "auflösg. alle Skalen  
 unmöglich"

$\Rightarrow$  Ström. wird durch ge-  
 mittelte Größen beschreiben

Reynolds Mittelung  
 mit: zeitl. Mittelung

Zerlegung nach Reynolds

$$f = \bar{f} + f'$$

mit

$$\bar{f} = \frac{1}{T} \int_t^{t+T} f(x, y, z, t) dt$$

zeitl. Mittelwert

$f'$ : Schwankungsteil

Erhaltungsgleichungen in Differenzform

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$$

$$\frac{\partial (\rho \bar{u})}{\partial t} + \frac{\partial (\rho \bar{u} \bar{u})}{\partial x} + \frac{\partial (\rho \bar{u} \bar{v})}{\partial y} + \frac{\partial (\rho \bar{u} \bar{w})}{\partial z} = - \frac{\partial \rho \bar{u}'}{\partial t} + \frac{\partial \rho \bar{u} u'}{\partial x} + \frac{\partial \rho \bar{u} v'}{\partial y} + \frac{\partial \rho \bar{u} w'}{\partial z}$$

$$\frac{\partial (\rho \bar{v})}{\partial t} + \frac{\partial (\rho \bar{v} \bar{u})}{\partial x} + \frac{\partial (\rho \bar{v} \bar{v})}{\partial y} + \frac{\partial (\rho \bar{v} \bar{w})}{\partial z} = - \frac{\partial \rho \bar{v}'}{\partial t} + \frac{\partial \rho \bar{v} u'}{\partial x} + \frac{\partial \rho \bar{v} v'}{\partial y} + \frac{\partial \rho \bar{v} w'}{\partial z}$$

$$\frac{\partial (\rho \bar{w})}{\partial t} + \frac{\partial (\rho \bar{w} \bar{u})}{\partial x} + \frac{\partial (\rho \bar{w} \bar{v})}{\partial y} + \frac{\partial (\rho \bar{w} \bar{w})}{\partial z} = - \frac{\partial \rho \bar{w}'}{\partial t} + \frac{\partial \rho \bar{w} u'}{\partial x} + \frac{\partial \rho \bar{w} v'}{\partial y} + \frac{\partial \rho \bar{w} w'}{\partial z}$$

Kontin. glg.:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \bar{u})}{\partial x} + \frac{\partial (\rho \bar{v})}{\partial y} + \frac{\partial (\rho \bar{w})}{\partial z} = 0$$

$$\frac{\partial \rho u'}{\partial t} + \frac{\partial (\rho \bar{u} u')}{\partial x} + \frac{\partial (\rho \bar{v} u')}{\partial y} + \frac{\partial (\rho \bar{w} u')}{\partial z} = 0$$

$$0 = \frac{\partial \rho u'}{\partial t} + \frac{\partial (\rho \bar{u} u')}{\partial x} + \frac{\partial (\rho \bar{v} u')}{\partial y} + \frac{\partial (\rho \bar{w} u')}{\partial z} =$$

x- Impulsglg.:

$$\frac{\partial (\rho \bar{u} s)}{\partial t} + \frac{\partial (\rho \bar{u} \bar{u} s)}{\partial x} + \frac{\partial (\rho \bar{u} \bar{v} s)}{\partial y} + \frac{\partial (\rho \bar{u} \bar{w} s)}{\partial z} =$$

$$- \frac{\partial (\rho \bar{u} s')}{\partial t} + \frac{\partial (\rho \bar{u} u' s')}{\partial x} + \frac{\partial (\rho \bar{u} v' s')}{\partial y} + \frac{\partial (\rho \bar{u} w' s')}{\partial z}$$

$$= \frac{\partial (\rho \bar{u} s)}{\partial t} + \frac{\partial (\rho \bar{u} \bar{u} s)}{\partial x} + \frac{\partial (\rho \bar{u} \bar{v} s)}{\partial y} + \frac{\partial (\rho \bar{u} \bar{w} s)}{\partial z} -$$

$$\frac{\partial (\rho \bar{u} s')}{\partial t} + \frac{\partial (\rho \bar{u} u' s')}{\partial x} + \frac{\partial (\rho \bar{u} v' s')}{\partial y} + \frac{\partial (\rho \bar{u} w' s')}{\partial z}$$

$$\frac{\partial (\rho \bar{u} s)}{\partial t} + \frac{\partial (\rho \bar{u} \bar{u} s)}{\partial x} + \frac{\partial (\rho \bar{u} \bar{v} s)}{\partial y} + \frac{\partial (\rho \bar{u} \bar{w} s)}{\partial z} +$$

$$\frac{\partial (\rho \bar{u} s')}{\partial t} + \frac{\partial (\rho \bar{u} u' s')}{\partial x} + \frac{\partial (\rho \bar{u} v' s')}{\partial y} + \frac{\partial (\rho \bar{u} w' s')}{\partial z}$$

oder

$$\frac{\partial}{\partial x} (s^2 \overline{u u'}) , \frac{\partial}{\partial y} [s (\overline{u v'} + u' \overline{v'})] ,$$

$$\frac{\partial}{\partial z} [s (\overline{u w'} + u' \overline{w'})] , \frac{\partial \overline{p}}{\partial x} , \Delta^2 \overline{u'} \rightarrow 0$$

$y$ -Imp.,  $z$ -Imp. ebenso  $\Rightarrow$

$$\frac{\overline{m}}{\partial x} + \frac{\partial \overline{w}}{\partial y} = 0$$

$$s (\overline{u} \frac{\partial \overline{m}}{\partial x} + \overline{v} \frac{\partial \overline{m}}{\partial y} + \overline{w} \frac{\partial \overline{m}}{\partial z}) = - \frac{\partial \overline{p}}{\partial x} + \nu \Delta^2 \overline{u}$$

$$- \left[ \frac{\partial \overline{m}}{\partial x} s + \frac{\partial \overline{v}}{\partial y} s + \frac{\partial \overline{w}}{\partial z} s \right]$$

$$s (\overline{u} \frac{\partial \overline{v}}{\partial x} + \overline{v} \frac{\partial \overline{v}}{\partial y} + \overline{w} \frac{\partial \overline{v}}{\partial z}) = - \frac{\partial \overline{p}}{\partial y} + \nu \Delta^2 \overline{v}$$

$$- \left[ \frac{\partial \overline{v}}{\partial x} s + \frac{\partial \overline{v}}{\partial y} s + \frac{\partial \overline{v}}{\partial z} s \right]$$

$$s (\overline{u} \frac{\partial \overline{w}}{\partial x} + \overline{v} \frac{\partial \overline{w}}{\partial y} + \overline{w} \frac{\partial \overline{w}}{\partial z}) = - \frac{\partial \overline{p}}{\partial z} + \nu \Delta^2 \overline{w}$$

$$- \left[ \frac{\partial \overline{w}}{\partial x} s + \frac{\partial \overline{w}}{\partial y} s + \frac{\partial \overline{w}}{\partial z} s \right]$$

die gemittelten Produkte ist.  
 turb. Schwankungsgrößen gesehen  
 oder turb. od. Reynoldschen  
 Spannungstensor.

$$\begin{pmatrix} \overline{\sigma'_{xx}} & \overline{\sigma'_{xy}} & \overline{\sigma'_{xz}} \\ \overline{\sigma'_{xy}} & \overline{\sigma'_{yy}} & \overline{\sigma'_{yz}} \\ \overline{\sigma'_{xz}} & \overline{\sigma'_{yz}} & \overline{\sigma'_{zz}} \end{pmatrix} = - \rho \begin{pmatrix} \overline{u'v'} & \overline{u'w'} & \overline{v'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix}$$

$-\rho \overline{p'g'}$ : Reynolds od. scheinbare  
 Spannungsen

turb. Strög.:

Gesamtspg. = visk. Spg. + turb. Spg.

i. a. gilt: turb. Spg.  $\gg$  visk. Spg.

Vereinfach. d. Bsp. Erhaltungsglgn.  
 mittels

- ebene Strög.  $\overline{w} = 0, \frac{\partial}{\partial z} \rightarrow 0$

- Grenzschicht:  
 $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y} \left[ \nu \frac{\partial \bar{u}}{\partial y} - \overline{u'v'} \right]$$

Grenzschichtgleich. turb. Strömung.

zusätzlich Turbulenz:  $-\rho \overline{u'v'}$

Kandbedingungen analog zur lamin. Strömung.

Schließung d. Gleichungssystems durch d. Prandtl'sche Mischungsweghypothese

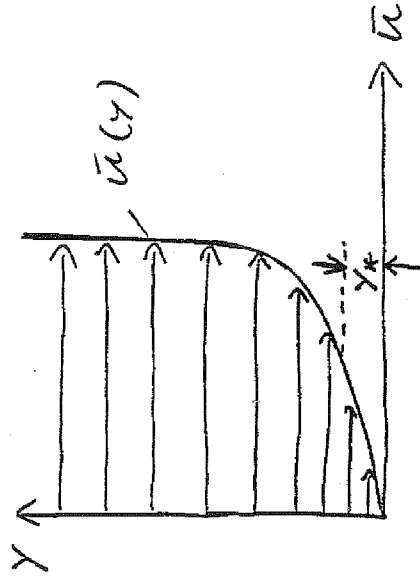
$$\tau_t = -\rho \overline{u'v'} = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} = \tau_t \frac{d\bar{u}}{dy}$$

$l = f(\text{charakterist. Länge})$

Turbulente Plattengrenzschicht

Annahme: Grenzschicht ist turb. ab  $x = 0$

$u(y)_{GS}$  vgl.  $u(y)_{\text{Rohr}}$



zähe Unterschicht  $y < y^*$

$$\frac{\bar{u}}{u_*} = \frac{u_* y}{\nu} \quad \text{mit. } u_* = \sqrt{\frac{\tau_w}{\rho}}$$

log. Schicht  $y^* < y$

$$\frac{\bar{u}}{u_*} = 2.5 \ln \frac{y u_*}{\nu} + C$$

univ. Wandgesetz



Rechnung: Was unmittelbare  
Wandgesetz gilt nicht in  
Wandnähe ( $y < y_*$ ).

äußere Grenzschichtbereich

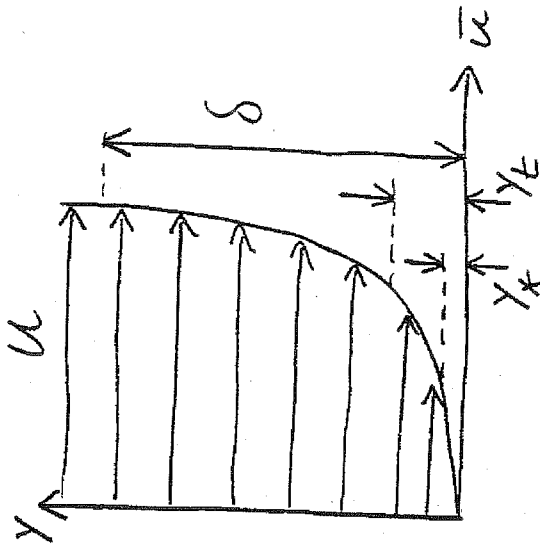
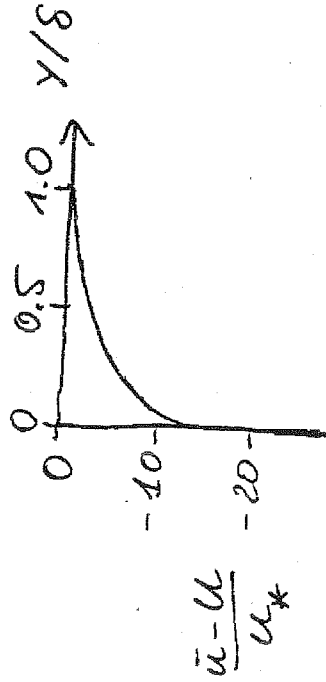
$y_* \ll y \leq \delta$  Abnahme der  
turb. Spg. u.

$u - \bar{u} \sim \tau_w$  bzw.  $u_* \Rightarrow$

$\frac{u - \bar{u}}{u_*} = f\left(\frac{y}{\delta}\right)$

$f\left(\frac{y}{\delta}\right) : ?$  Annahme: univ. Wand:  
gesetz gilt im Außenbereich

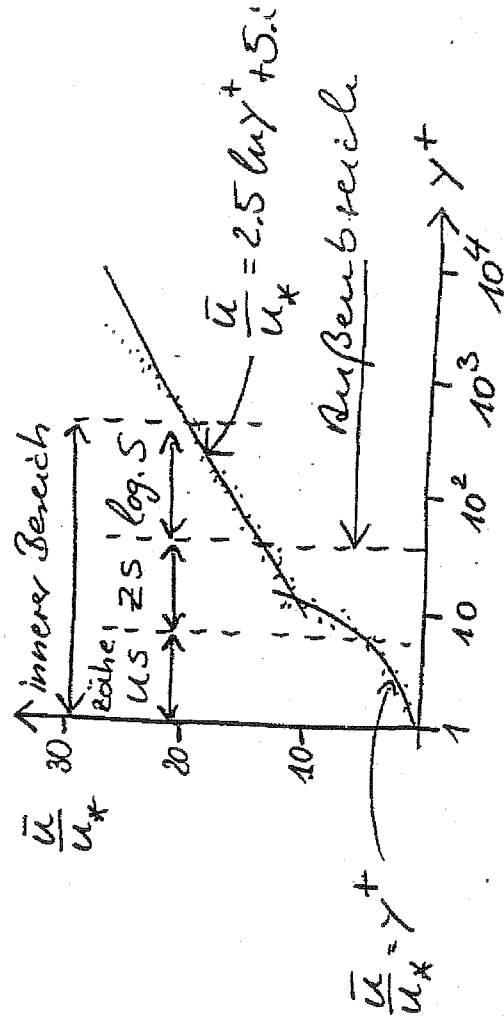
nach Prandtl  $f\left(\frac{y}{\delta}\right) = \frac{1}{\kappa} \ln \frac{y}{\delta}$



Zusammenfassung:

- $0 \leq y \leq y_*$
- $y_* < y < y_t$
- $y_t \leq y \leq \delta$

nahe Wandschicht  
Übergangsschicht  
Außenbereich



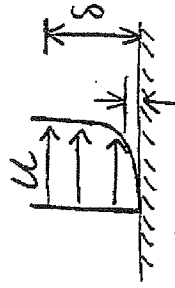
Herleitung des logarith. Gesetzes  
mittels Dimensionsbetrachtung.

- Geschw. profil in Wandnähe

$$\bar{u} = \bar{u}(u_*, \gamma, \nu)$$

$\Pi$ -Theorem : 4 Var., 2 Ref. dim.

$$\Rightarrow \frac{\bar{u}}{u_*} = f\left(\frac{u_* \gamma}{\nu}\right) = f(\gamma_+)$$



zähe Unterschicht:

$$\tau_w = \rho \frac{d\bar{u}}{dy}$$

$$\bar{u} = \gamma \tau_w / \rho = \gamma \rho u_*^2 / \rho$$

$$\frac{\bar{u}}{u_*} = \gamma \frac{u_*}{\nu} = \gamma_+$$

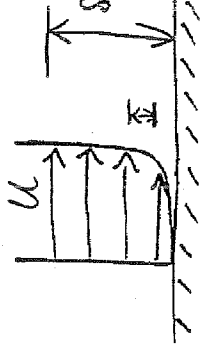
Außenbereich:



$$\bar{u} - u \sim u_*$$

$$\frac{\bar{u} - u}{u_*} = g\left(\frac{\gamma}{s}\right) = g(f)$$

Gebiet zw. Innen- u. Außenbereich



Anpassung v.

$$\frac{d\bar{u}}{dy} \text{ für } \gamma_+ \rightarrow \infty$$

$$f \rightarrow 0$$

$$\Rightarrow f(\gamma_+) \text{ und } g(f)$$

Innenbereich:

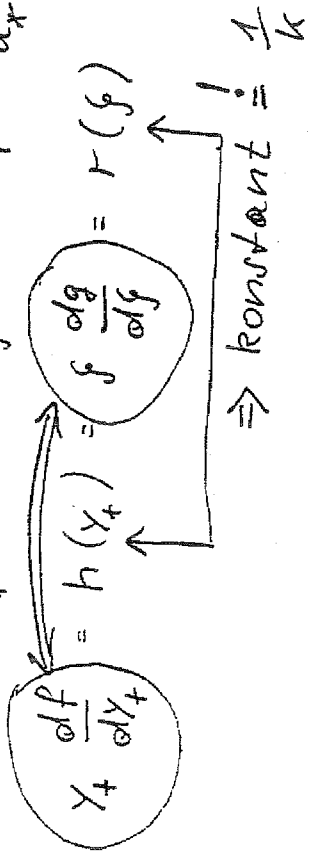
$$\frac{d\bar{u}}{dy} = u_* \frac{df}{d\gamma_+} \frac{d\gamma_+}{dy} = \frac{u_*^2}{\nu} \frac{df}{d\gamma_+}$$

Außenbereich:

$$\frac{d\bar{u}}{dy} = u_* \frac{dg}{df} \frac{df}{d\gamma} = \frac{u_*}{s} \frac{dg}{df}$$

$$\frac{d\bar{u}}{dy} \Big|_{\text{Innen}} = \frac{d\bar{u}}{dy} \Big|_{\text{Außen}}$$

$$\frac{u_*^2}{\nu} \frac{df}{dy_+} = \frac{u_*}{\delta} \frac{dg}{df} \Big| \cdot \frac{y}{u_*}$$



$$\Rightarrow f(y_+) = \frac{1}{k} \ln y_+ + \alpha$$

$$g(y_+) = \frac{1}{k} \ln(y_+) + \beta$$

$$\text{Exp.: } k = 0.4, \alpha = 5.5, \beta = -1.0$$

$$\Rightarrow y_+ \text{ groß: } \frac{\bar{u}}{u_*} = 2.5 \ln y_+ + 5.5$$

(log. Gesetz)

$$y_+ \text{ klein: } \frac{\bar{u} - u}{u_*} = 2.5 \ln(y_+) - 1.0$$

(Defektverteilg.)

Näherung von  $\bar{u}(y)$  mittels

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{1/n}$$

Vgl. mit Exp.  $\rightarrow n = 7 \Rightarrow$

Verdrängungsdicke  $\delta_1$

$$\delta_1 = \delta \int_0^1 \left(1 - \frac{\bar{u}}{U}\right) d\left(\frac{y}{\delta}\right) = \frac{\delta}{8}$$

Impulsverlustdicke  $\delta_2$

$$\delta_2 = \delta \int_0^1 \frac{\bar{u}}{U} \left(1 - \frac{\bar{u}}{U}\right) d\left(\frac{y}{\delta}\right) = \frac{7}{72} \delta$$

!  $\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{1/7}$  nicht gültig bei  $y=0$ !

$\Rightarrow \bar{u}$  aus d. turb. Rohrströmung

$$\text{Mit } \lambda = \frac{0.316}{4 \sqrt{\frac{u_*}{\nu}}} \text{ in } \lambda = \frac{8 \bar{u}}{8 \bar{u}^2}$$

und  $\frac{\bar{u}}{U} = 0.8$  folgt

XXI. Turbulent flow through pipes

Table 20.1. Ratio of mean to maximum velocity in pipe flow in terms of the exponent of the velocity distribution, according to eqn. (20.6)

$n$	5	6	7	8	9	10
$\bar{u}/u_m$	0.791	0.817	0.837	0.852	0.863	

a. Relation between law of friction and velocity distribution

The equation for the velocity distribution (20.6) is related to Blasius's law of friction in eqn. (20.5) and this relation, first discovered by L. Prandtl [5], is of fundamental importance in the theory of turbulent flow; it allows us to draw conclusions from pipe experiments which are valid for the flat plate [52]; use of them will be made in Chap. XXII.

On substituting the value of  $\lambda$  from eqn. (20.5) into eqn. (20.4) we obtain the following expression for the shearing stress at the wall:

$$\tau_w = 0.03325 \bar{u}^2 \rho^{1/2} \mu^{1/4} R^{-1/4} \nu^{-1/4}$$

Introducing the radius  $R$  instead of the diameter  $d$  it is necessary to divide the numerical factor in the above equation by  $2^{1/4} = 1.19$ . Thus we obtain:

$$\tau_w = 0.0276 \bar{u}^2 \rho^{1/2} \mu^{1/4} R^{-1/4} \nu^{-1/4}$$

where  $\rho = \rho_0$  is the density of the fluid,  $\mu$  is the dynamic viscosity,  $R$  is the radius of the pipe,  $\nu$  is the kinematic viscosity. If we split  $\rho = \rho_0$  into  $\rho = \rho_0 \epsilon^{1/4}$  and  $\epsilon = \nu^2 / \mu^2$ , we obtain

$$\left(\frac{\tau_w}{\rho_0 \nu}\right)^4 = \frac{0.003325^4}{\epsilon} \left(\frac{\bar{u}}{\nu}\right)^4 \quad \text{or} \quad \left(\frac{\tau_w}{\rho_0 \nu}\right)^4 = 0.003325^4 \left(\frac{\bar{u}}{\nu}\right)^4$$

and if we eliminate the mean velocity  $\bar{u}$  with the aid of the maximum velocity  $U_m$  by putting  $\bar{u}/U_m = 0.8$  which, as seen from Table 20.1, corresponds approximately to an exponent  $n = 7.1$ ,  $n$  to a Reynolds number  $R = 10^4$ , we have

$$\frac{\tau_w}{\rho_0 \nu} = 8.74 \left(\frac{\bar{u}}{\nu}\right)^{1/4}$$

It is now natural to assume that this equation is valid for any wall distance  $y$ , and not only for the pipe axis (wall distance  $y = R$ ). Hence, we obtain from eqn. (20.5)

$$n = 8.74 \left(\frac{\bar{u}}{\nu}\right)^{1/4}$$

The preceding argument shows that the  $i$ -th power velocity-distribution law can be derived from Blasius's resistance formula. It has already been shown before that such a law agrees with experiment over a certain range of Reynolds numbers, and it is seen that there exists a relation between Blasius's law of friction and the  $n$ -th-power velocity-distribution law. Introducing the abbreviations  $u^+ = \bar{u} \nu / \mu$  and

XXI. Turbulent flow through pipes

For future reference we now propose to write down an expression for the friction velocity  $u_*$  from eqn. (20.10). We obtain

$$u_* = 0.150 \nu \left(\frac{\bar{u}}{\nu}\right)^{1/4}$$

and

$$\tau_w = \rho u_*^2 = 0.0225 \bar{u}^2 \rho \nu^{1/2}$$

or

$$c_f = \frac{\tau_w}{\rho U_m^2} = 0.045 \left(\frac{\bar{u}}{U_m}\right)^{1/2}$$

where  $c_f$  denotes the local skin-friction coefficient. This relation, which is equivalent to the one in eqn. (20.5), is known as Blasius's law of skin friction in pipe flow. This relation will be used later.

c. Universal velocity-distribution laws for very large Reynolds numbers

The fact that the exponent in the law of pipe resistance as well as in the expression for velocity distribution decreases with increasing Reynolds numbers suggests that both must tend asymptotically to some expressions which are valid for very high Reynolds numbers and which must contain the logarithm of the independent variable, as is the limit of a polynomial for very small values of the exponent. A detailed examination of experimental results for very large Reynolds numbers shows that such logarithmic laws do, in fact, exist. Physically such asymptotic laws are characterized by the fact that laminar friction becomes completely negligible compared with turbulent friction. The great advantage of such logarithmic laws, as compared with the  $1/n$ -th-power law, consists in their being asymptotic expressions for very large Reynolds numbers; they may, therefore, be extrapolated to arbitrarily large values beyond the range covered by experiment. On the other hand, when the  $1/n$ -th-power laws are used the value of the exponent  $n$  changes, as the range of Reynolds numbers is extended.

Such an asymptotic logarithmic law has already been given in eqn. (19.33) for the case of flow along a flat plate. It was deduced from Prandtl's equation (19.7) for turbulent shearing stress under the assumption that the mixing length is proportional to the distance from the wall,  $l = \kappa y$ , and was valid for small wall distances  $y$ . This equation has the form:

$$\phi = A_1 \ln \eta + D_1 \quad (20.13)$$

F. M. Reichardt [53] indicated a refined expression for the velocity distribution. It covers the whole range of distances, from the wall of the pipe at  $y = 0$  to the centre-line at  $y = R$ . It is also valid for the laminar sub-layer, to which eqn. (20.13) does not apply. It is also valid in the neighbourhood of the centre-line, where measured velocity-distribution curves (Fig. 19.20) are well reproduced by the formula. This universal velocity-distribution law was deduced with the aid of theoretical considerations and very careful measurements of the turbulent mixing coefficient  $\lambda$  defined by eqn. (19.1). Compare also a paper by W. Stabbe [54].

b. Relation between law of friction and velocity distribution

$u^+/\nu = \eta$ , which were already used in eqns. (19.31) and (19.32), we can transform eqn. (20.10) to

$$\phi = 8.74 \eta^{1/4}$$

Thus we have once more deduced eqn. (19.32) which was first obtained from considerations of similarity, except that the numerical values of the constants  $C$  and  $D$  which then remained undetermined, are now known from the law of pipe friction equation (20.11) has been compared with J. Nikuradse's experiments in Fig. 20.1. It is seen that the  $i$ -th-power law agrees well with experiment up to the Reynolds number of about  $R = 100,000$ . No better agreement can be expected because Blasius's equation (20.5) from which it was derived is valid only to this limit, Fig. 20.1.

In order to obtain better agreement it would be necessary to introduce a small component into Blasius's equation, say  $\lambda$  or  $\lambda'$  instead of  $\lambda$ . Performing corresponding calculations it is found that the exponent  $n$  in the velocity-distribution law would have to be replaced by  $n + \lambda'$ , i.e., respectively, in agreement with measured values the relation  $\tau_w/\rho = C \bar{u}^2 \nu^{1/2}$  has been obtained as curve (a) in Fig. 20.4, and it is seen that it does, in fact, reproduce the experimental values at higher Reynolds numbers with a good measure of agreement, but that the fit is inferior at lower Reynolds numbers.

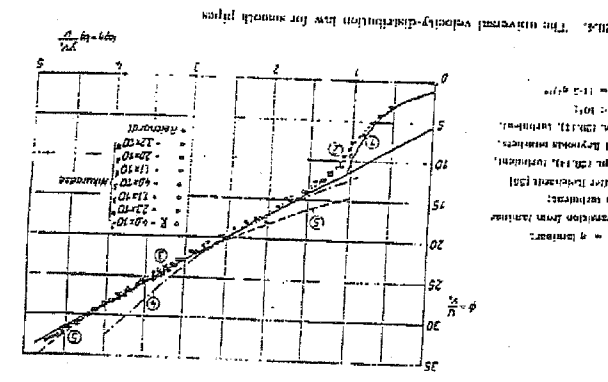


Fig. 20.4. The universal velocity-distribution law for smooth pipes.  $\lambda = C \bar{u}^2 \nu^{1/2}$  and the following numerical values:  $\lambda = 0.003325$ ,  $\lambda' = 0.0001$ ,  $\lambda'' = 0.0001$ ,  $\lambda''' = 0.0001$ ,  $\lambda^{(4)} = 0.0001$ ,  $\lambda^{(5)} = 0.0001$ ,  $\lambda^{(6)} = 0.0001$ ,  $\lambda^{(7)} = 0.0001$ ,  $\lambda^{(8)} = 0.0001$ ,  $\lambda^{(9)} = 0.0001$ ,  $\lambda^{(10)} = 0.0001$ .

with  $\lambda_1 = 1/\kappa$  and  $D_1 = - (1/\kappa) \cdot \ln \beta$  as free constants. We shall apply this equation without change to pipe flow. Comparing it with the measurements performed by J. Nikuradse, as shown by curve (b) in Fig. 20.4, it is seen that excellent agreement is obtained not only for points near the wall but for the whole range up to the axis of the pipe. The numerical values of the constants are found to be

$$\lambda_1 = 2.5, \quad D_1 = 5.5$$

This gives the following values of  $\kappa$  and  $\beta$ :

$$\kappa = 0.4, \quad \beta = 0.11$$

Hence the universal velocity-distribution law for very large Reynolds numbers has the form:

$$\phi = 2.5 \ln \eta + 5.5 \quad (\text{smooth})$$

By a reasoning similar to the one given in the preceding section it is possible to arrive at a corresponding universal asymptotic resistance coefficient formula from the above equation (20.14), being one for turbulent flow, is valid only in regions where the laminar shearing stress can be neglected in comparison with the turbulent stress. In the immediate neighbourhood of the wall, where the turbulent shearing stress increases to zero and laminar stresses predominate, deviations from the law must be expected. L. Reichardt [54, 55] extended this kind of measurement to include very small distances from the wall in a flow in a channel. Curve (c) in Fig. 20.4 represents the transition from the laminar sub-layer (cf. Sec. XVII c) to the turbulent boundary layer. The curve showed by (1) in the above diagram corresponds to laminar flow for which  $\tau_w = \rho u_*^2$ . With  $\tau_w = \rho u_*^2$  we obtain  $u^+ = 1.72 \eta^{1/2}$  or

$$\phi = \eta \quad (\text{laminar})$$

From this it can be seen that for values  $\eta \ll 5$  the contribution from turbulent friction may be completely neglected compared with laminar friction. In the range  $5 < \eta < 70$  both contributions are of the same order of magnitude, whereas for  $\eta \gg 70$  the laminar contribution is negligible compared with turbulent friction. Thus:

$$\frac{\eta}{5} < 5: \quad \text{purely laminar friction}$$

$$5 < \eta < 70: \quad \text{laminar-turbulent friction}$$

$$\eta > 70: \quad \text{purely turbulent friction}$$

In the following equations  $\eta$  denotes the natural logarithm and  $\ln$  the logarithm to base 10.

$$\frac{U}{u_*} = 8.74 \left( \frac{u_* x}{\nu} \right)^{1/4}$$

$$\Rightarrow u_* = 0.15 U^{7/8} \left( \frac{\nu}{x} \right)^{1/8} \quad \text{bzw.}$$

$$\tau_w = \rho u_*^2 = 0.0225 \rho U^{7/4} \left( \frac{\nu}{x} \right)^{1/4}$$

v. Karman'sche Integralbereich.

$$\frac{d\delta_2}{dx} = \frac{\tau_w}{\rho U^2} = 0.0225 \left( \frac{\nu}{U\delta} \right)^{1/4}$$

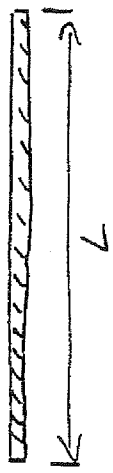
$$\rightarrow \frac{7}{72} \frac{d\delta}{dx} = \frac{d\delta_2}{dx}$$

$$\Rightarrow \boxed{\frac{\delta(x)}{x} = \frac{0.37}{\left( \frac{Ux}{\nu} \right)^{1/5}} = \frac{0.37}{Re_x^{1/5}}}$$

D.h.  $\delta \sim x^{4/5}$  turb. Grenz.

$\delta \sim x^{1/2}$  lam. Grenz.

Reibungskraft auf eine Seite



$$D = \int_0^L \tau_w dx = 0.036 \rho U^2 L \frac{\nu^{1/5}}{(Re_L)^{1/5}}$$

$$\Rightarrow D \sim L^{4/5} \quad \text{bzw.} \quad D \sim U^{3/5}$$

$$[\text{lam.}: D \sim L^{1/2} \quad \text{bzw.} \quad D \sim U^{3/2}]$$

$$C_D = \frac{D}{\frac{\rho}{2} U^2 L} = \frac{0.072}{(Re_L)^{1/5}} = \frac{0.074}{(Re_L)^{1/5}} \quad \text{Exp.}$$

allgemein:  $C_D = f(Re_L, \nu/L)$

bzw.  $C_D = f(\nu/L)$  für

$Re_L$  sehr groß!

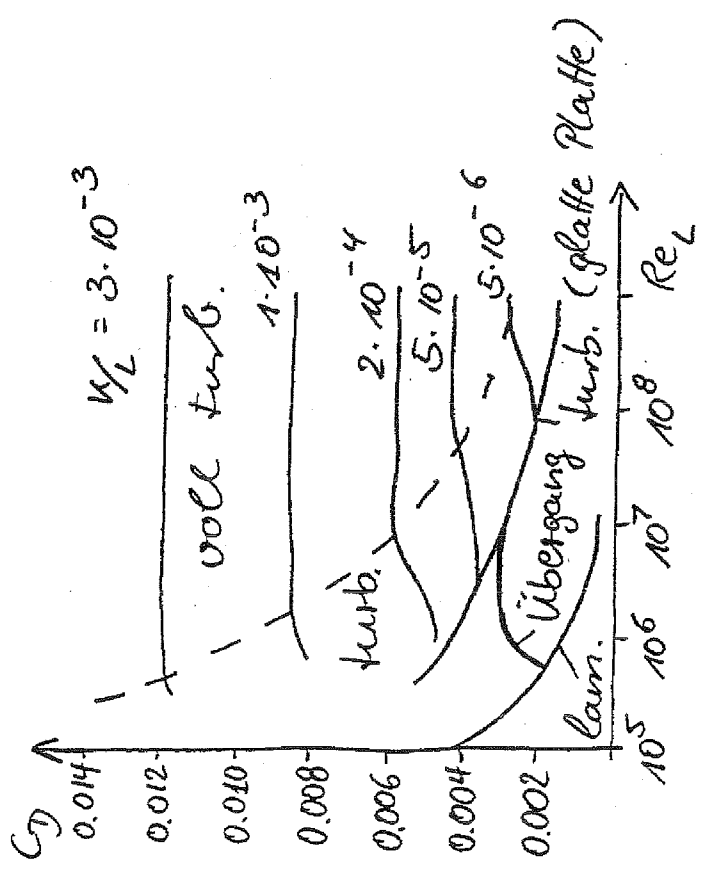
# Bemerkungen zur Turbulenz

vierteilhaft:

- Wärmeübergang zw. Wand und Fluid
  - Klimaanlagen
  - Kessel im Kraftwerk
- Vermischung von Fluiden
- Rauch aus Schornsteinen

lam. Strömungswiderstand:

- Druckverlust d. Rohrström.
  - Widerstand eines Tragflügels
- jedoch: Vermeidg. v. Ablös. in turb. Ström. => größter Auftrieb als mit Ablös.



Übergangsbereich:

$$C_D = \frac{0.074}{\sqrt{Re_L}} - \frac{1700}{Re_L}$$

glatte Platte:

$$C_D = \frac{0.455}{(\log Re_L)^{2.58}}$$

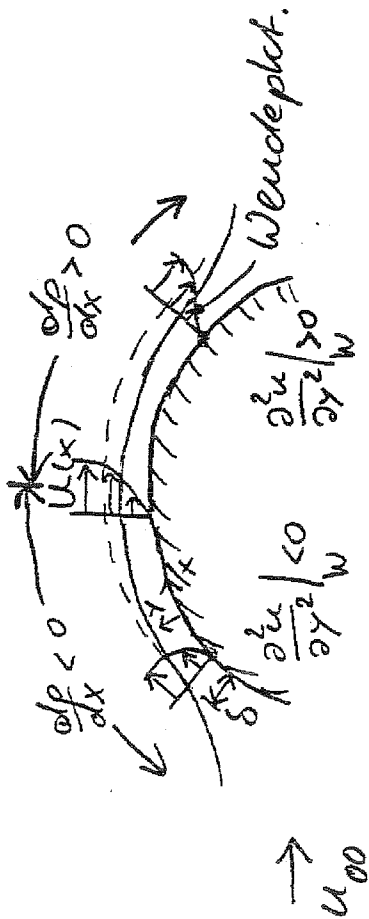
voll turb. Gebiet:

$$C_D = [1.89 - 1.62 \log(\nu/L)]^{-2.5}$$

# Grenzschichtablösung

## Einfluß des Druckgradienten

ebene Platte:  $\frac{dp}{dx} = 0$



Auswirkungen von  $\frac{dp}{dx}$  auf  $u(y)$ ?

Grenzschichtglg.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

weiterhin

$$\frac{\partial f}{\partial x} = \frac{dp}{dx} = -\rho u \frac{du}{dx}$$

Wand  $y=0$ :  $u=v=0$

$$\Rightarrow \frac{dp}{dx} = \rho \frac{\partial^2 u}{\partial y^2}$$

$$\frac{dp}{dx} < 0 \text{ (beschl. Ström.)}: \left. \frac{\partial^2 u}{\partial y^2} \right|_{\text{Wand}} < 0$$

nahe d. Grenzschichtrand  $\delta$  gilt

$\frac{\partial u}{\partial y} > 0$ ; mit

$$\left. \frac{\partial u}{\partial y} \right|_{y=\delta} \rightarrow 0 \text{ folgt } \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=\delta} < 0$$

$\Rightarrow$  kein Vorzeichenwechsel  $v$ .

$\frac{\partial^2 u}{\partial y^2}$  in  $0 \leq y \leq \delta$  (i.a.)

$\frac{df}{dx} > 0$  (verzögerte Strög.):  $\frac{\partial^2 u}{\partial y^2} \Big|_{\text{Wand}} > 0$

$\Rightarrow$  Vorzeichenwechsel zw.  $0 \leq y \leq \delta$

$\rightarrow u(y)$  hat Wendepunkt, in

dem  $\frac{\partial^2 u}{\partial y^2} = 0$

Bemerkg.: ebene Platte  $\frac{\partial^2 u}{\partial y^2} \Big|_{y=0} = 0$

$\frac{df}{dx} > 0 \Rightarrow \delta(x)$  wächst deutl. an,

denn

$$v(x, y) = - \int_0^y \frac{\partial u}{\partial x} dx = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$-\frac{\partial u}{\partial x} \uparrow \Rightarrow v \uparrow \Rightarrow \delta \uparrow$

Mit  $\frac{df}{dx} > 0$  ist i.a. d. Ablösung

der Strömung verbunden.

## Ablösung

$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } 0 \leq y \leq \delta$$

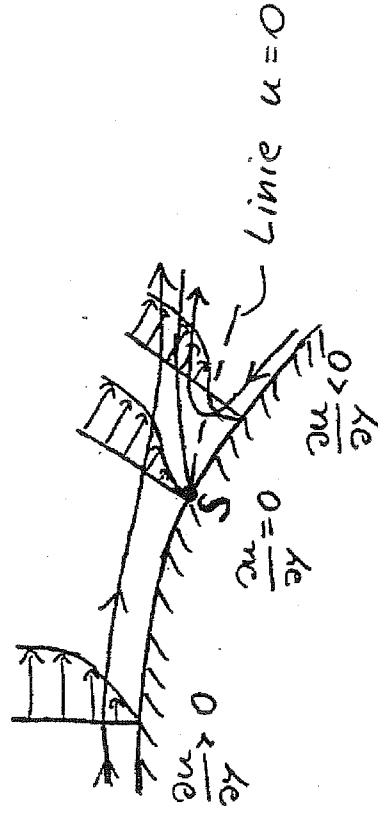
$$\Rightarrow \frac{\partial u}{\partial x} \Big|_{y=\gamma_w+\epsilon} < 0 \quad \text{aufgrund v. } \frac{df}{dx} > \epsilon$$

sofern  $\frac{df}{dx} > 0$  groß, folgt

$$u_1(x_1, \gamma_w+\epsilon) > 0, \quad u_2(x_1+\epsilon, \gamma_w+\epsilon) < 0$$

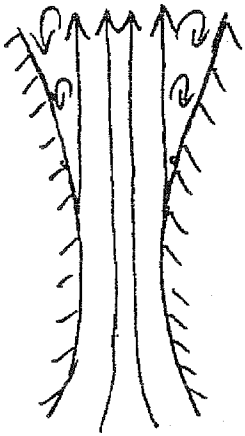
$\rightarrow$  Strömungsablösung

$$\text{Ablösepunkt: } \frac{\partial u}{\partial y} \Big|_{\text{Wand}} = 0$$





Ablösung bei externen und internen Strömungen



Die Grenzschichtgln. sind bis  $x = x_s$  gültig; für  $x > x_s$  Grenzschichtannahmen i.a. ungültig;

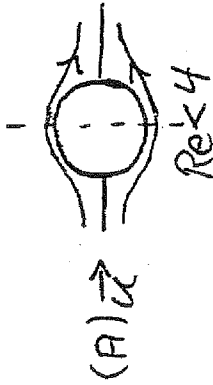
Strömung über einen Kreisylinder.

Ablösg. =  $f(Re)$  vor allem bei stumpfen Körpern.

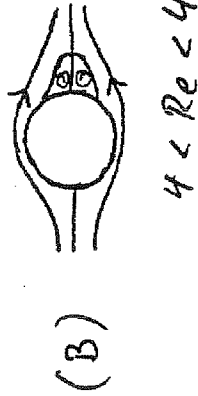
Analyse f. d. Kreisylinder

$$Re = \frac{u \cdot d_s}{\nu}$$

$Re < 4$  :  $c_D \sim Re^{-1}$ , keine Ablös.



$4 < Re < 40$  : Bilobig. zweier an-  
liegenden Wirbel



für  $Re > 40$  : Nachlauf wird instabil; Geschw. ist period. in  $t$  un in  $x$  (für  $x > x_s$ ).

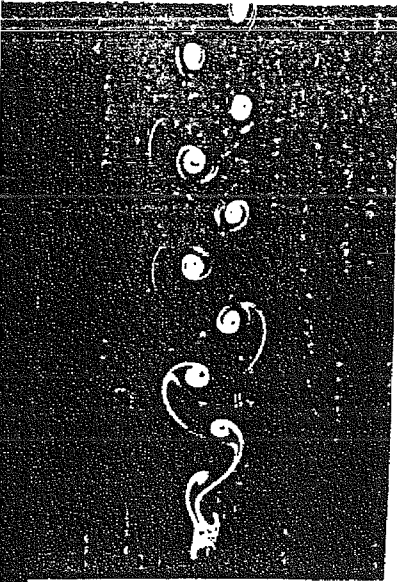


94. Kármán vortex street behind a circular cylinder at  $R=140$ . Water is flowing at  $1.4 \text{ cm/s}$  past a cylinder of diameter  $1 \text{ cm}$ . Integrated streaklines are shown by electrolytic precipitation of a white colloidal smoke, illuminated

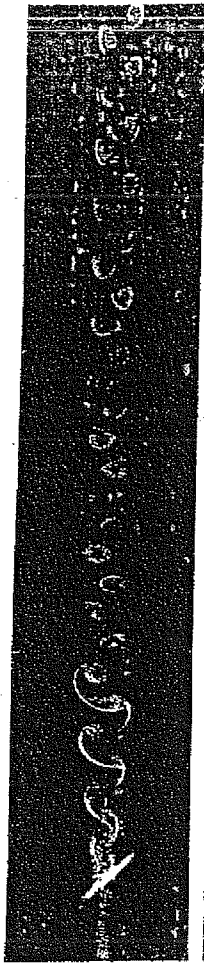
by a sheet of light. The vortex sheet is seen to grow in width downstream for some diameters. Photograph by Sada-toshi Tanieda



95. Kármán vortex street behind a circular cylinder at  $R=200$ . This photograph, made using a different fluid (and in another country) happens to have been timed so as to resemble remarkably the flow pattern in the upper picture. A thin sheet of tobacco smoke is introduced upstream in a low-turbulence wind tunnel. Photograph by Gary Koopman



96. Kármán vortex street behind a circular cylinder at  $R=105$ . The initially spreading wake shown opposite develops into the two parallel rows of staggered vortices that von Kármán's inviscid theory shows to be stable when the ratio of width to streamwise spacing is  $0.28$ . Streaklines are shown by electrolytic precipitation in water. Photograph by Sada-toshi Tanieda

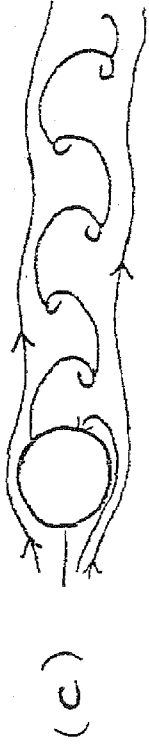


97. Smoke at various levels in a vortex street. A smoke filament in air shows, at a Reynolds number of  $100$ , both shear layers (top photograph), only one shear layer (middle), and the irrotational flow below the wake (bottom). Zdravkovich 1969



98. Kármán vortices in absolute motion. Here the camera moves with the vortices rather than the cylinder. The streamline pattern closely resembles the inviscid one calculated by von Kármán. The flow is visualized by particles floating on water. Photograph by R. Wille, from Wörlé 1973. Reproduced, with permission, from the Annual Review of Fluid Mechanics, Volume 5, © 1973 by Annual Reviews Inc.

$80 < Re < 200$  : 2 versetzte Wirbel  
im Nachlauf



$$80 < Re < 200$$

→ v. Kármán'sche Wirbelstr.

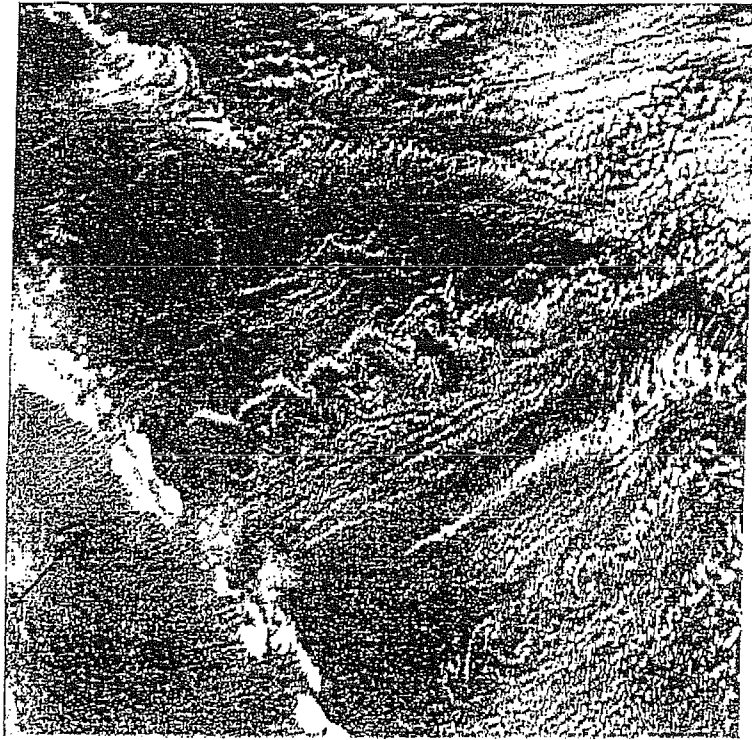
$$u_{\text{vor, ab}} < u$$

$40 < Re < 80$  : anliegende Wirbel  
nicht Teil d.  
Wirbelstraße.

$Re > 80$  : Wirbel → Oszillation  
d. Zylinders

Strouhal Zahl :  $St = f \frac{d}{u}$   
diml. Frequenz d. abgehenden

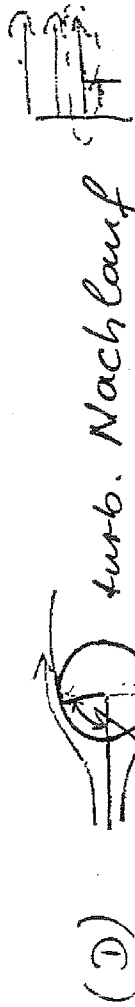
Wirbel :  $St = 0.21$  (Experiment)



von Kármán'sche Wirbel=  
straße(n), die infolge der  
Strömung über die Gipfel  
eines Bergkammes entstehen

$u$  klein,  $u$  klein  $\rightarrow f$  im hörbar. Bereich

$Re < 3 \cdot 10^5$ : Ablösg. bei  $\approx 82^\circ$ ,  
Grenzschicht ist lam.



$82^\circ$   
 $Re < 3 \cdot 10^5$

im Nachlauf:  $p \approx konst < p_{\infty}$

$3 \cdot 10^5 < Re < 3 \cdot 10^6$ : Grenzschicht ist  
turbulent, Ablösg. bei  
 $\approx 125^\circ$

$\Rightarrow$  Nachlauf wird  
schmäler

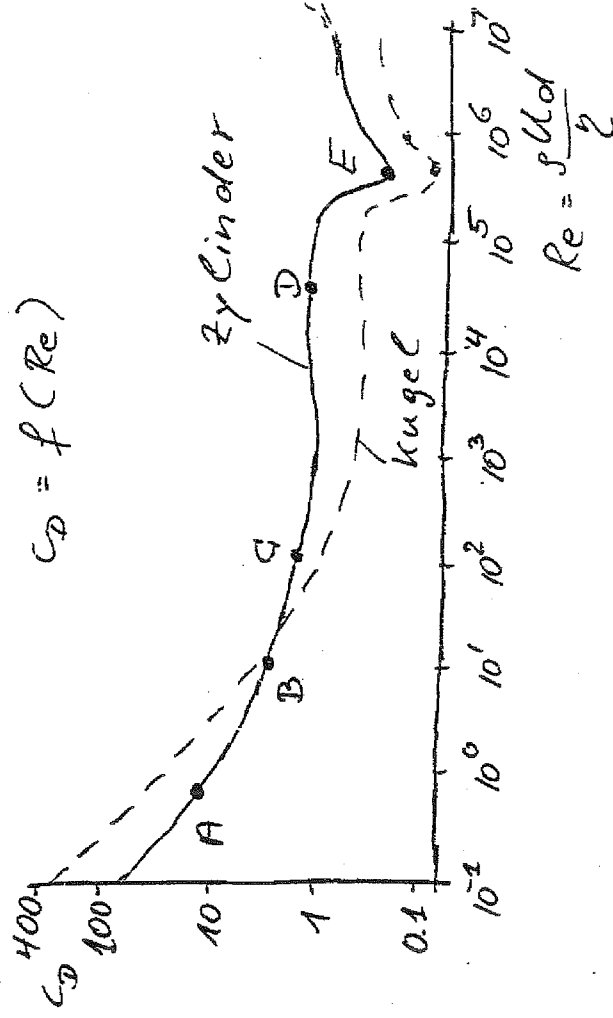


$125^\circ$   
 $Re > 3 \cdot 10^5$

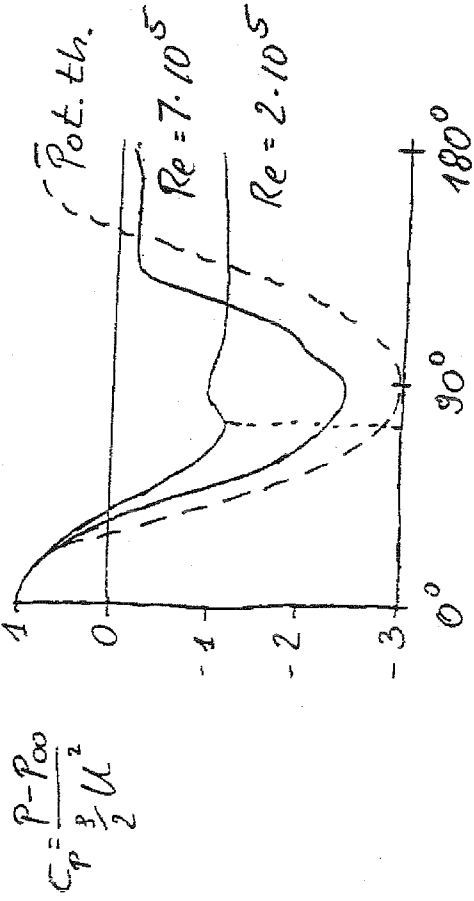
$Re > 3 \cdot 10^6$ :  $\theta_S < 125^\circ$

$\Rightarrow C_D$  steigt

$Re_{krit} = f(\text{Turb. in d. Anström, Oberflächenrauigkeit})$



$c_p = f(\theta)$ , Zylinder



Strömung über eine Kugel

Übergang  $2D \rightarrow 3D \Rightarrow$  deutliche  
Unterschiede

z.B.: keine reguläre Wirbelstr.

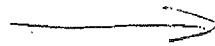
$Re < 130$ : anliegender Wirbel =  
Ring

$Re > 130$ : Oszillationen beginnen  
verzerrte Wirbelschleifen  
gehen ab;

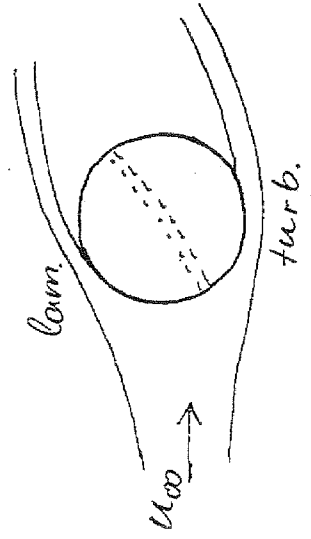
Verhalten d. Grenzschicht wie beim  
Kreiszyliner;

$Re_{krit} \approx 5 \cdot 10^5$  (Kugel)

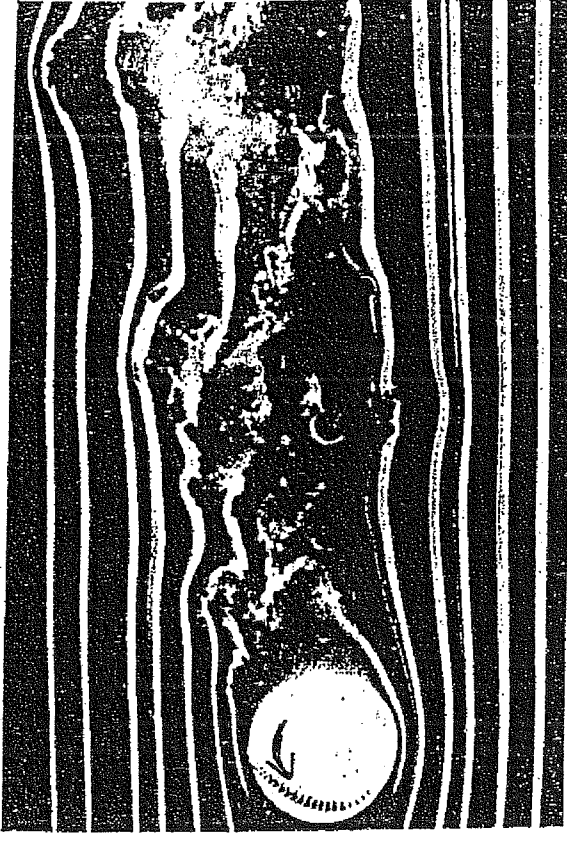
Transition lam. - turb.  $\Rightarrow c_D$  sinkt.



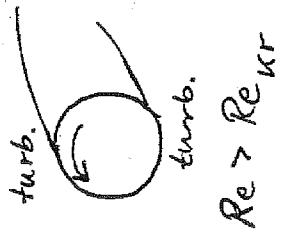
Strömung über einen  
Kricketball;



$Re \approx 10^5$   
↓ Richtung d.  
Seitenkr.



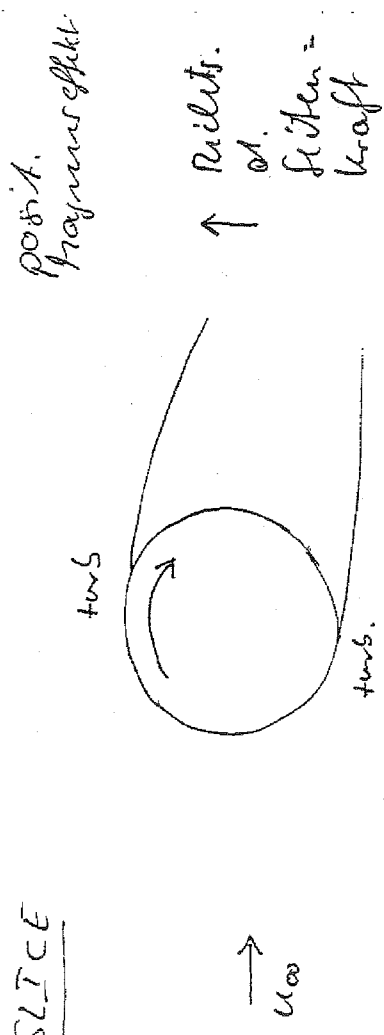
Strömung über einen rotierenden  
Baseball.



↓ Richtung d.  
Seitenkr.

# Tennisball

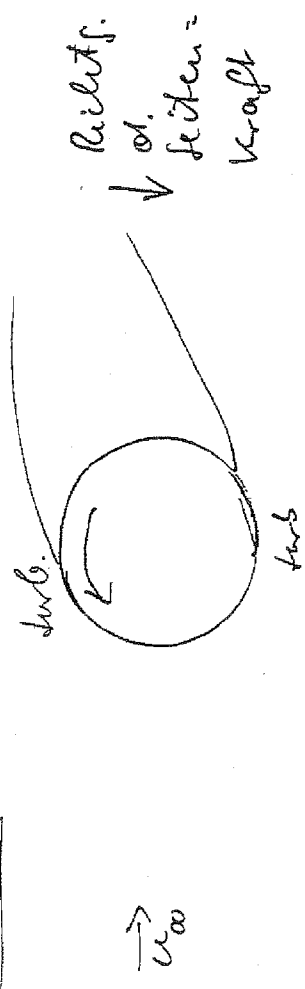
## SLICE



$$Re > Re_{kr}$$

→  $\begin{matrix} \text{down} \\ \text{sub.} \end{matrix} \downarrow$  Richtf. od. Seitenkraft  
neg. Magnus effekt!  
 $Re < Re_{kr}$

## TOPSPIN



$$Re > Re_{kr}$$

# Mehrphasenströmung

Strömungen, in denen die Wechselwirkung zw. d. Phase (fest, flüssig, gasförmig) eine wesentliche Rolle spielt

BSP.: Meteorologie

Klimatisierung

Verbrennung

Verfahrenstechnik

Beschreibungsform

2phasenströg. (Fluid, Partik.)

→ Lagrange: Bahnverfolgung d. Partikel

→ Euler: Änderung v. Partikelklass.

## Kompressible Strömungen

bisher: dichtebeständige Fluide

im folgenden: dichteveränderliche

bzw. kompressible Fluide

→ Gasdynamik

Beschränkung: stationäre, 1-D,  
reibungsfrei, kompr. Strömu.  
ideales Gas

## Schallgeschwindigkeit

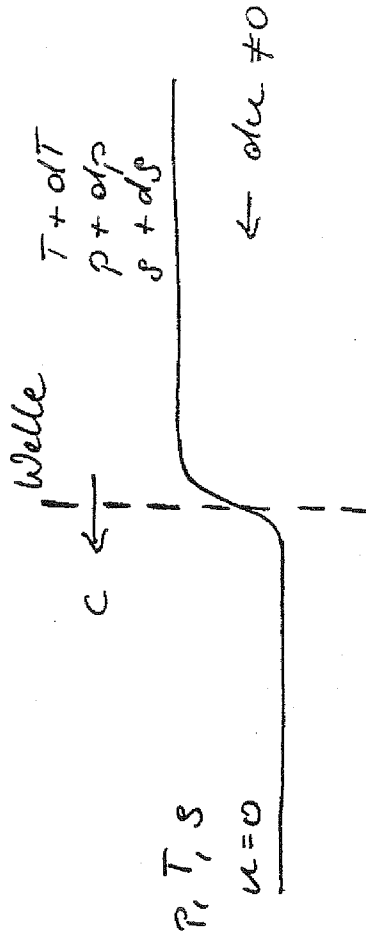
inkompressible Strög.: p-Störung  
überall, sofort meßbar

kompressible Strög.: p-Störung  
breitet sich als elast.  
Welle aus

akustische od. Schallwelle:

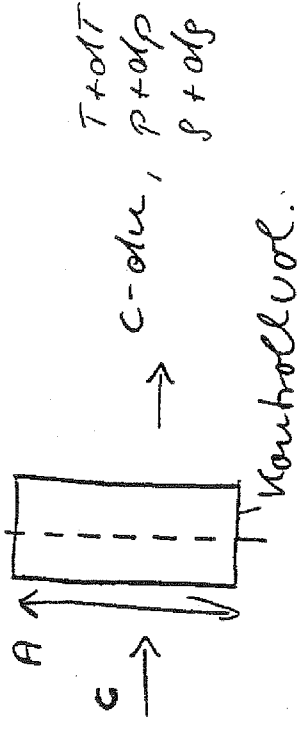
Wellen mit infinitesimaler  
Amplitude

Betr. der Schallgeschwindigkeit:



instat. Probl. f. ruhenden Beobach.  
stat. Probl. durch Übertrag. mit  
Geschw.  $c$  in entgegengesetzter Richtung

stat. Welle



Kontrollvolum.



in über A:

$$A_{sc} = A(p + dp)(c - du)$$

$$= A_{sc} + A c dp - A_{sc} du$$

$- A dp du$   
 vernachlässigbar

$$\Rightarrow du = c \frac{ds}{s}$$

Kompression:  $ds > 0 \rightarrow du > 0$

Expansion:  $ds < 0 \rightarrow du < 0$

Impuls:

$$A_{sc}(c - du) - A_{sc}c = pA - (p + dp)A$$

$$A(p + dp)(c - du) dp = s c du$$

$$\Rightarrow c^2 = \frac{dp}{ds}$$

Amplitude d. Welle ist infinites.  
 $\Rightarrow$  isentrope Zustandsänderg. d. Teilchen

$$\rightarrow c^2 = \left(\frac{dp}{ds}\right)_{s = \text{konst.}}$$

+ Isentropenbeziehung  $p/s^\gamma = \text{konst.}$

+ Gasgleichung  $p = sRT$

$$\rightarrow \left(\frac{dp}{ds}\right)_{s = \text{konst.}} = \frac{p}{s} \gamma = \gamma RT$$

bzw.

$$c = \sqrt{\gamma \frac{p}{s}} = \sqrt{\gamma RT}$$

Luft, 15°C,  $c = 340 \frac{\text{m}}{\text{s}}$

" , 20°C,  $c = 343 \frac{\text{m}}{\text{s}}$

Kompressibilitäts effekte von Bedeutung?

Antwort mittels  $M = \frac{u}{c}$

Kontig. (1D):  $u \frac{\partial s}{\partial x} + s \frac{\partial u}{\partial x} = 0$

inkompressibel, sofern

$$u \frac{\partial s}{\partial x} \ll s \frac{\partial u}{\partial x}$$

bzw.

$$\frac{\Delta s}{s} \ll \frac{\Delta u}{u}$$

Qualgeschw.:

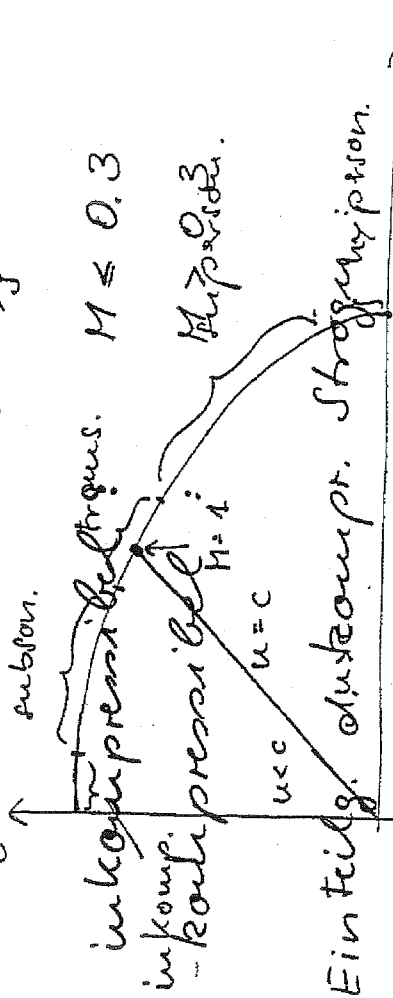
$$\Delta p \approx c^2 \Delta s$$

Euler (1D):

$$u \Delta u \approx \frac{\Delta p}{\rho}$$

$$\Rightarrow \frac{\Delta s}{s} \approx \frac{u^2}{c^2} \frac{\Delta u}{u} = M^2 \frac{\Delta u}{u}$$

D.h.  $M^2 < 1$  Strög.  $\frac{\Delta s}{s}$  sehr klein

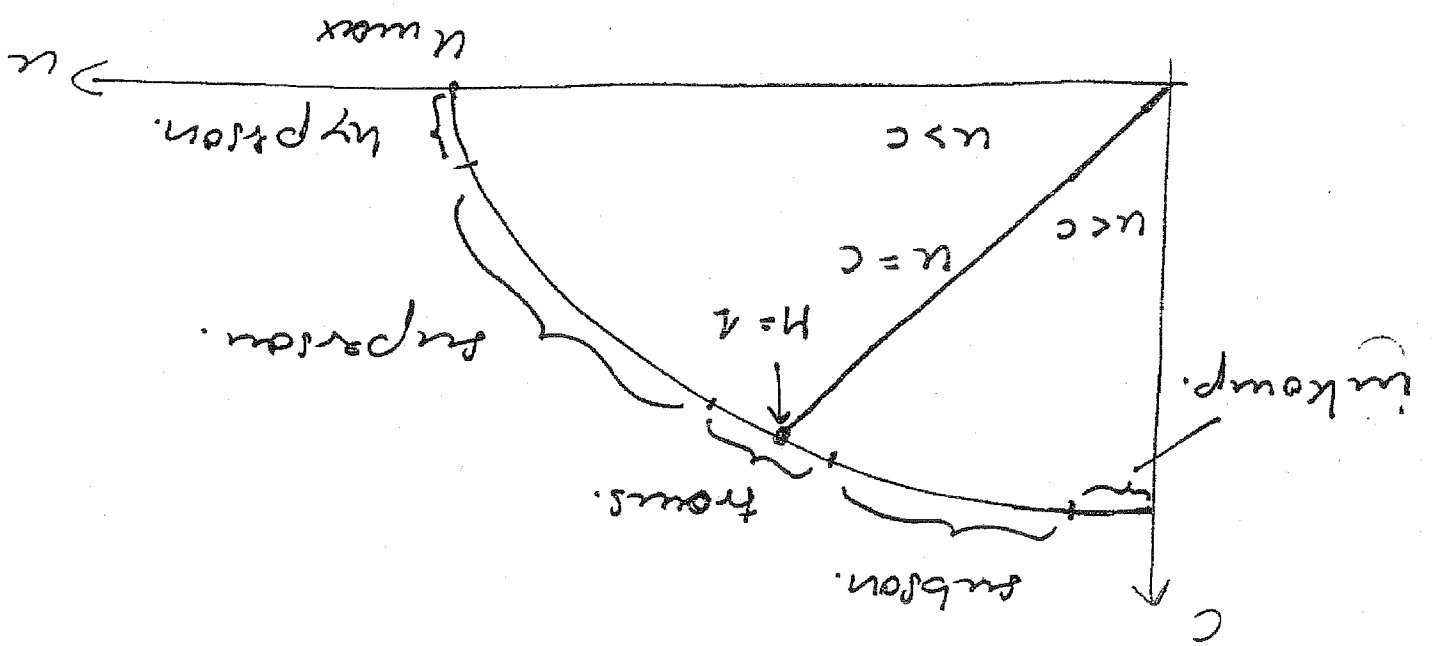


$0.3 < M < 1$  : subsonische Strög.

$0.8 < M < 1.2$  : transonische Strög.

$1 < M < 3$  : supersonische Strög.

$M > 3$  : hypersonische Strög.



$$\Rightarrow \frac{\Delta S}{S} \approx \frac{c^2}{u^2} \frac{\Delta u}{u} = H^2 \frac{\Delta u}{u}$$

D.h.  $H^2 \ll 1$  Shog.  $\frac{\Delta S}{S}$  sehr klein

inkomprimibel :  $M \leq 0.3$

kompromibel :  $M > 0.3$

Einheitsg. d. Komp. Shog. :

$0.3 < M < 1$  : subsonische Shog.

$0.8 < M < 1.2$  : transonische Shog.

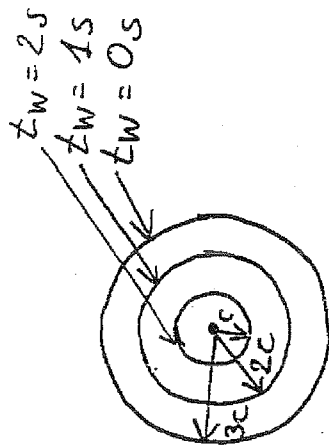
$1 < M < 3$  : supersonische Shog.

$M > 3$  : hypersonische Shog.

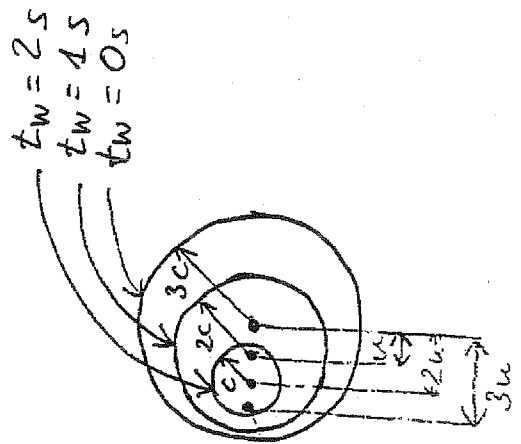
sofern  $M > 1$  ist das Ausbreitungsgebiet v. Störungen begrenzt  
 → Machkegel

Druckwelle wird bei  $t = t_w$  initiiert

Radius  $r = (t - t_w) c$

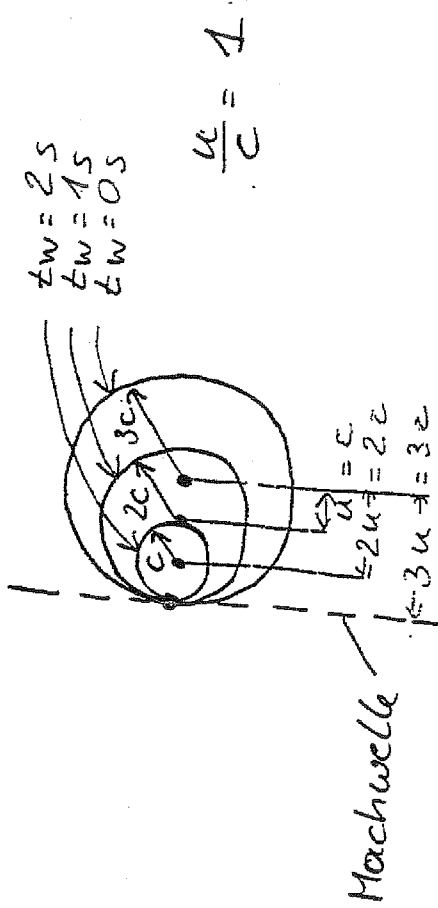


$u = 0$

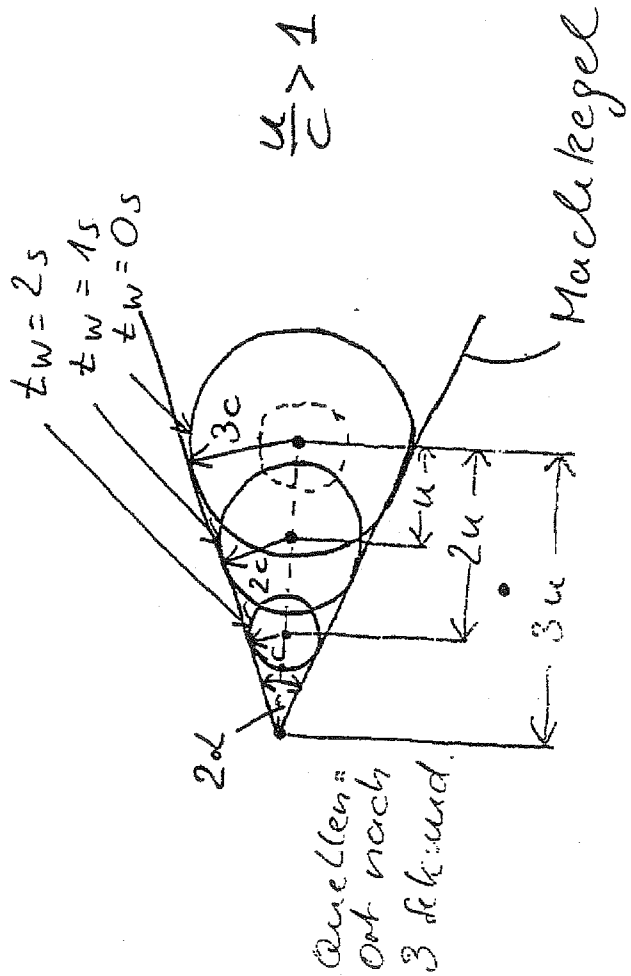


$\frac{u}{c} < 1$

Ort nach 3 Sekunden



$\frac{u}{c} = 1$



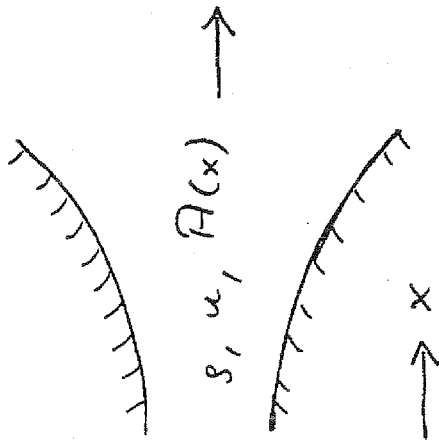
$\frac{u}{c} > 1$

Ort nach 3 Sekunden

Öffnungswinkel des Machkegels

$$\sin \alpha = \frac{c}{u} = \frac{1}{M}$$

Flächen-Geschwindigkeits-Bezieh.



isentrop. Ström. durch ein Rohr

Kontinuitätsgl.:

$$\rho u A = \text{konst.}$$

Differenziation

$$\frac{d(\rho u A)}{\rho u A} = \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler (1D):

$$u du = - \frac{dp}{\rho} = - \frac{dp}{\rho} \frac{ds}{s} = - c^2 \frac{ds}{s}$$

$$\Rightarrow \frac{ds}{s} = - M^2 \frac{du}{u} \quad \frac{ds}{s} = - \frac{u^2 du}{c^2 u}$$

bzw.

$$\frac{du}{u} = - \frac{dA}{A} \frac{1}{1-M^2}$$

Flächen-Geschw.-Bezieh.

Daraus folgen interessante Konsequenzen der Kompressibilität

besgl. der Auswirkungen von Änderungen v.  $A(x)$  auf  $u(x)$ .

# Ruhe- und kritische Größen

Ruhezustad: isentrope Verzögerung.

auf  $\vec{v} = 0$ ;

dient als Referenz:  
Zustad.

Energiegl. im Ruhezustad.

$$\frac{v_0^2}{2} + h_0 = h + \frac{u^2}{2} \quad \frac{T_0}{T} = ?$$

id. Gas  $h = c_p T$

$$c_p T_0 = c_p T + \frac{u^2}{2}$$

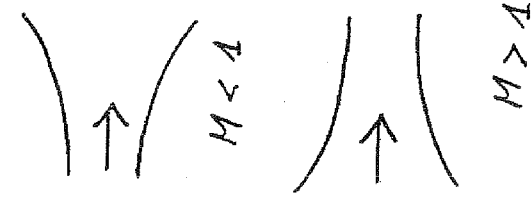
$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2c_p T} = 1 + \frac{u^2}{2} \frac{\gamma - 1}{\gamma R T} c_p^2$$

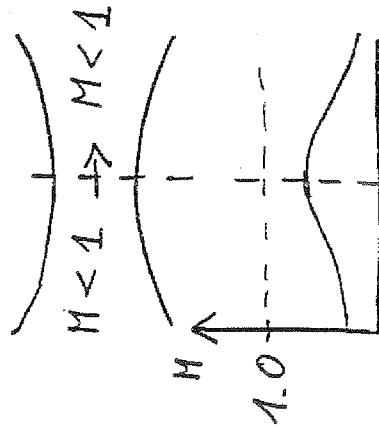
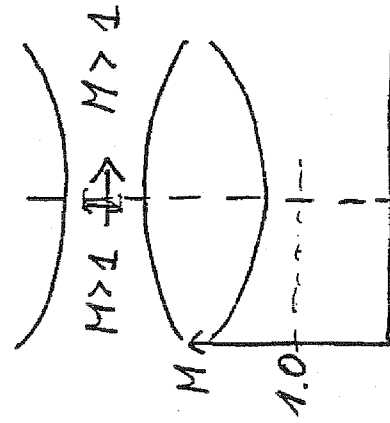
$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \quad \leftarrow$$

$$\frac{T_0}{T} = f(M)$$

Diffusor  
 $dp > 0$   
 $du < 0$



Düse  
 $dp < 0$   
 $du > 0$



Isenropen oder Riemannsche  $\Rightarrow$

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}} = g(M)$$

$$\frac{s_0}{s} = \left(\frac{T_0}{T}\right)^{\frac{1}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{1}{\gamma-1}} = l(M)$$

in adiabater Strömung  $\left\{ h + \frac{u^2}{2} = \text{konst.} \right\}$   
gilt f. d. Ruhegrößen

$$h_0 = \text{konst.}, T_0 = \text{konst.},$$

$$c_0 = \sqrt{\gamma R T_0} = \text{konst.}$$

Bezieh. zwischen  $u$  und  $p$ :

$$c_p T + \frac{u^2}{2} = c_p T_0$$

$$u^2 = \frac{2\gamma}{\gamma-1} R T_0 \left(1 - \frac{T}{T_0}\right)$$

$$u = \left\{ \frac{2\gamma}{\gamma-1} \frac{p_0}{s_0} \left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] \right\}^{\frac{1}{2}}$$

Vakuum  $p \rightarrow 0 \Rightarrow u \rightarrow u_{\max}$

$$u_{\max} = \left[ \frac{2\gamma}{\gamma-1} \frac{p_0}{s_0} \right]^{\frac{1}{2}}$$

Nachzahl als  $f(p)$ :

$$M = \left[ \frac{u^2}{c^2} \right]^{\frac{1}{2}} = \left[ \frac{u^2}{\gamma \frac{p}{s}} \right]^{\frac{1}{2}} = \left\{ \frac{2}{\gamma-1} \left[ \left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1 \right] \right\}^{\frac{1}{2}}$$

$$p \rightarrow 0 : M \rightarrow \infty$$

Konvergenz - divergente Düse: bei  
geeignet. Gegenbruch strömt man

im Halboverschritt. Zust. ( $M=1$ )  
wird als kritische Zust. bezeichnet.

Für den kritischen Zust.,  
ebenfalls ein Reibzust.,  
ersicht sich



$$c_p T_0 = c_p T^* + \frac{\gamma R T^*}{2} \quad ; \quad (\gamma = 1.4)$$

$$\frac{T^*}{T_0} = \frac{2}{\gamma + 1} = 0.833$$

$$\frac{P^*}{P_0} = \left( \frac{T^*}{T_0} \right)^{\frac{\gamma}{\gamma-1}} = \left( \frac{2}{\gamma+1} \right)^{\frac{\gamma}{\gamma-1}} = 0.528$$

$$\frac{S^*}{S_0} = \left( \frac{2}{\gamma+1} \right)^{\frac{1}{\gamma-1}} = 0.634$$

Bereueg.  $\frac{A^*}{A} = f(M, \gamma)$  mittels Kont. glg



$$\rho u A = \rho^* u^* A^*$$

$$\frac{A^*}{A} = \frac{\rho}{\rho^*} \frac{u}{u^*}$$

$$\frac{A^*}{A} = \frac{\rho}{\rho^*} \frac{S_0}{S^*} \frac{c}{c^*} \frac{u}{u^*}$$

$$\frac{A^*}{A} = \frac{\rho}{\rho^*} \frac{S_0}{S^*} \frac{c}{c^*} \frac{u}{u^*} \Rightarrow \frac{A^*}{A} = f(M) \quad f(\gamma)$$

$$\frac{S}{S_0} \frac{S_0}{S^*} = \frac{1}{\left[ \left( 1 + \frac{\gamma-1}{2} M^2 \right) \frac{2}{\gamma+1} \right]^{\frac{\gamma-1}{2}}}$$

$$\frac{c}{c_0} \frac{c_0}{c^*} \frac{u^*}{u^*} = \left( \frac{T}{T_0} \frac{T_0}{T^*} \right)^{\frac{1}{2}} = \frac{1}{\left[ \left( 1 + \frac{\gamma-1}{2} M^2 \right) \frac{2}{\gamma+1} \right]^{\frac{1}{2}}}$$

$$\frac{A^*}{A} = \frac{M}{\left[ \left( 1 + \frac{\gamma-1}{2} M^2 \right) \frac{2}{\gamma+1} \right]^{\frac{\gamma+1}{2(\gamma-1)}}} = f(M, \gamma)$$

$$\text{mit } \frac{S_0}{S^*} \left( \frac{T_0}{T^*} \right)^{\frac{1}{2}} = \frac{1}{\left( \frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

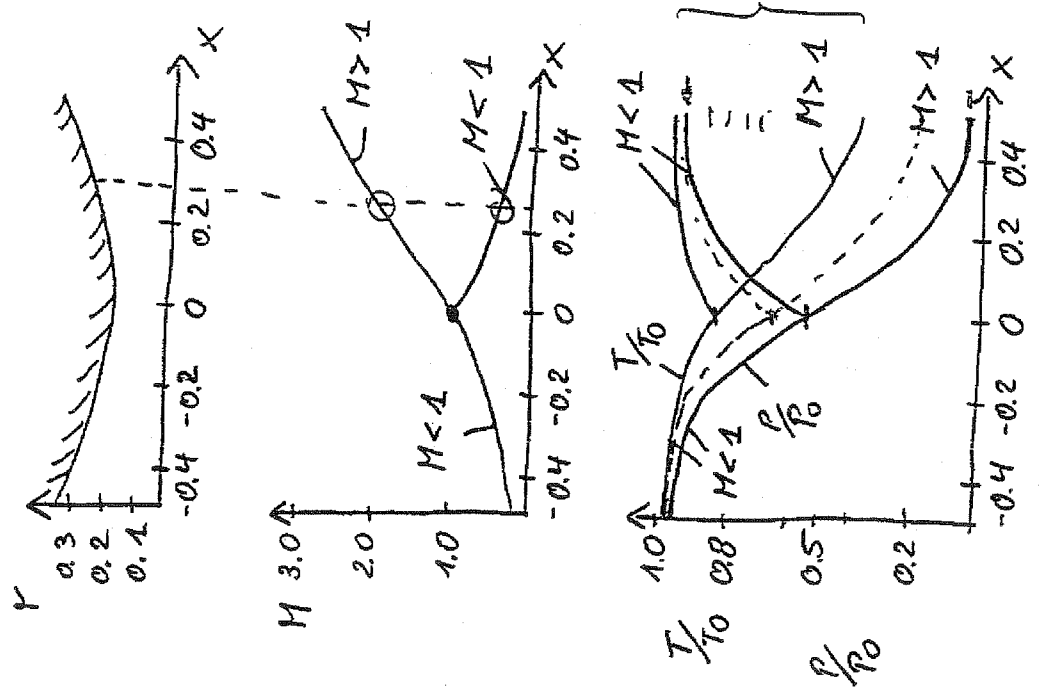
$$\frac{S}{S_0} \left( \frac{T}{T_0} \right)^{\frac{1}{2}} = \left( \frac{P}{P_0} \right)^{\frac{1}{\gamma}} \left( \frac{P}{P_0} \right)^{\frac{\gamma-1}{2\gamma}}$$

$$M = \left( \frac{2}{\gamma-1} \right)^{\frac{1}{2}} \left( \frac{P_0}{P} \right)^{\frac{\gamma-1}{2\gamma}} \left[ 1 - \left( \frac{P}{P_0} \right)^{\frac{\gamma-1}{\gamma}} \right]^{\frac{1}{2}}$$

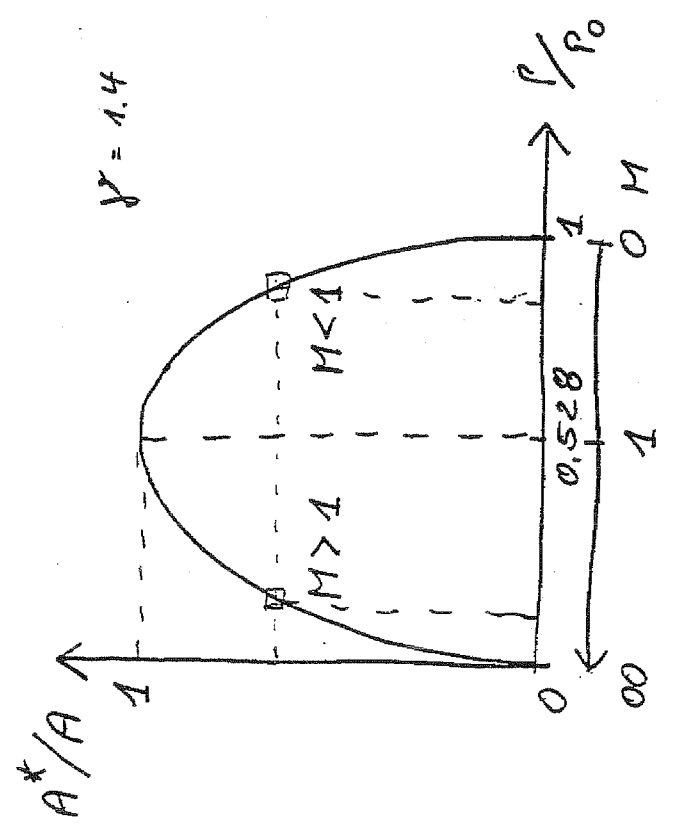
$$\Rightarrow \frac{A^*}{A} = \left( \frac{P}{P_0} \right)^{\frac{1}{\gamma}} \frac{\left[ 1 - \left( \frac{P}{P_0} \right)^{\frac{\gamma-1}{\gamma}} \right]^{\frac{1}{2}}}{\left[ \frac{2}{\gamma-1} \left( \frac{2}{\gamma+1} \right) \right]^{\frac{1}{2}}} = f \left( \frac{P}{P_0}, \gamma \right)$$

$A^*/A$  bekannt  $\rightarrow M, P/P_0, T/T_0$  Verlauf

z. Bsp.:  $r = \left( \frac{0.1 + x^2}{\pi} \right)^{1/2}$



$A^*/A \rightarrow 0$  für  $M \rightarrow 0$ , bzw.  $P/P_0 \rightarrow 1, P/P_0 \rightarrow \infty$   
 $A^*/A \rightarrow 1$  für  $M \rightarrow 1$  bzw.  $P/P_0 \rightarrow 0.528$



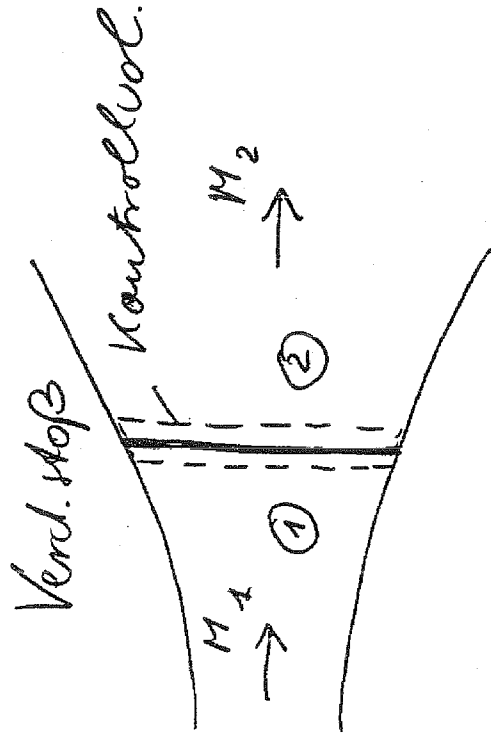
krit. Var. sind Ref. größen, wenn  $M=1$  Zustand isentrop erreicht wird.

# Der senkrechte Machstörungsstoß

Machstörungsstoß od. Stoßwelle ist eine Diskontinuität endl. Stärke,

über d. Stoß sind die Isentropenbeziehungen ungültig.

Zustände vor und hinter dem Stoß:



Sucht:  $\frac{u_2}{u_1}, \frac{p_2}{p_1}, \frac{T_2}{T_1}$  etc.!

Masse-, Impuls- u. Energieerhaltung ( $dA \approx 0$ )

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_2 u_2^2 - \rho_1 u_1^2 = p_2 - p_1$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

Impuls:

$$u_2 - u_1 = \frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = \frac{c_1^2}{\rho_1 u_1} - \frac{c_2^2}{\rho_2 u_2}$$

$$\gamma (u_2 - u_1) = \left( \frac{c_1}{u_1} \right)^2 u_1 - \left( \frac{c_2}{u_2} \right)^2 u_2 \quad (*)$$

Referenzstoß  $\rightarrow$  \* - Stoß:

$$\frac{u^2}{2} + \frac{c^2}{\gamma-1} = c^{*2} \frac{\gamma+1}{\gamma-1} \frac{1}{2} = \frac{c^* \gamma + \frac{c^*}{2}}{\gamma-1}$$

$$\Rightarrow \left( \frac{c^*}{u} \right)^2 = \left[ \left( \frac{c}{u} \right)^2 + \frac{\gamma-1}{2} \right] \frac{2}{\gamma+1}$$

$$\left( \frac{u}{c^*} \right)^2 = M^{*2} = \frac{\gamma+1}{(\gamma-1) + \frac{2}{M^2}}$$

$$M^* = f(M, \gamma)$$

$$\begin{aligned}
 M = 0 & : M^* = 0 \\
 M = 1 & : M^* = 1 \\
 M \rightarrow \infty & : \lim_{M \rightarrow \infty} M^{*2} = \frac{\gamma+1}{\gamma-1}
 \end{aligned}$$

$$\begin{aligned}
 M < 1 & : M^* < 1 \\
 M > 1 & : M^* > 1
 \end{aligned}$$

$\frac{c}{u} = f\left(\frac{c}{u}, \gamma\right)$  in Glg. (\*)

$$\Rightarrow (u_2 - u_1) \frac{\gamma+1}{2} = c^{*2} \left( \frac{1}{u_1} - \frac{1}{u_2} \right) \frac{\gamma+1}{2}$$

$$c^{*2} = u_1 u_2$$

$M_1^* M_2^* = 1$

$$\frac{s_2}{s_1} = \frac{u_1}{u_2} = \frac{u_1^2}{c^{*2}} = M_1^{*2} = \frac{(\gamma+1) M_1^2}{(\gamma-1) M_1^2 + 2} = f(M_1, \gamma)$$

Impuls:

$$\frac{p_2}{p_1} = 1 + \gamma M_1^2 \left( 1 - \frac{u_2}{u_1} \right) = 1 + \frac{2\gamma}{\gamma+1} (M_1^2 - 1) = g(M_1, \gamma)$$

$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2} \frac{\gamma M_1^2 + 1}{M_1^2} (M_1^2 - 1) = h(M_1)$$

$$M_2^2 = \frac{2 + (\gamma-1) M_1^2}{2\gamma M_1^2 - (\gamma-1)}$$

eng. Rankine-Hugoniot Beziehungen

Entropieänderg. =  $f(M_1, \gamma)$ :

$$T ds = \alpha dh - \frac{1}{\rho} dp \quad \text{ist.}$$

Integration: Erfolgreich

$$\begin{aligned}
 \frac{s_2 - s_1}{R} &= \frac{c_p}{R} \ln\left(\frac{T_2}{T_1}\right) - \ln\left(\frac{p_2}{p_1}\right) \\
 &= \ln \left[ \left(\frac{s_2}{s_1}\right)^{\frac{c_p}{R}} \left(\frac{p_2}{p_1}\right)^{-\frac{1}{\gamma-1}} \right] \\
 &= \ln \left[ \left(\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}\right)^{\frac{\gamma}{\gamma-1}} \left[ 1 + \frac{2\gamma}{\gamma+1} (M_1^2 - 1) \right]^{-\frac{\gamma}{\gamma-1}} \right] \\
 &= f(M_1, \gamma)
 \end{aligned}$$

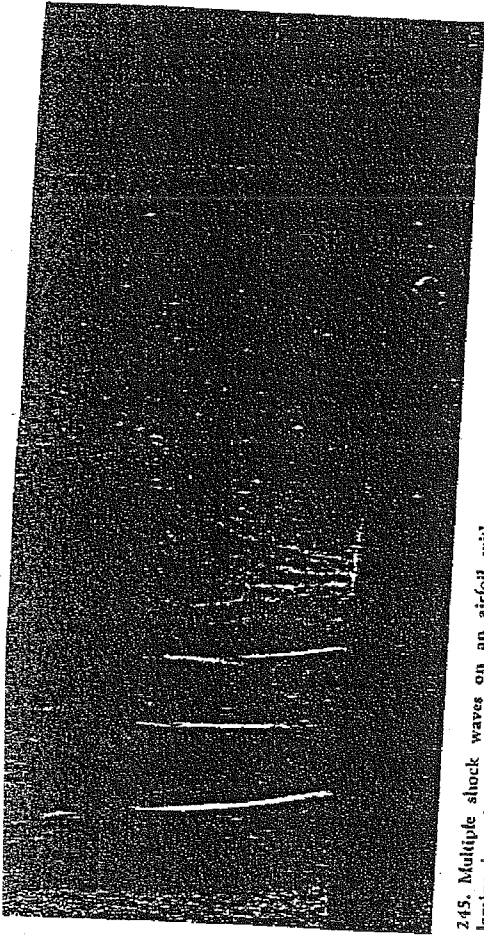
$s_2 - s_1 > 0$  : d.h. Stöße nur  
in superson. Strömen.

weil hier

$$s_2 - s_1 = s_{02} - s_{01} = c_p \ln \frac{T_{02}}{T_{01}} - R \ln \frac{p_{02}}{p_{01}}$$

mit  $T_0 = \text{konst}$  :  $p_{02} < p_{01}$

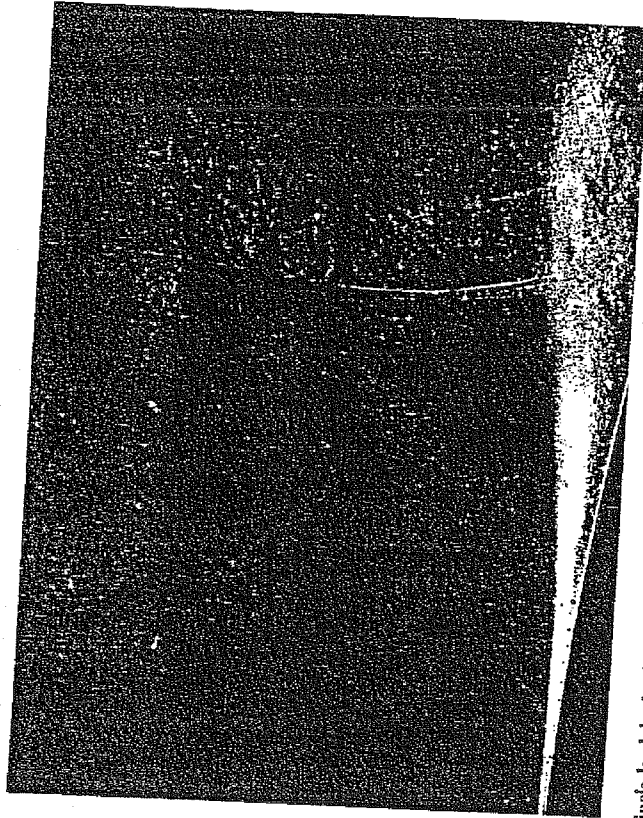
bzw.  $s_{02} < s_{01}$ .



245. Multiple shock waves on an airfoil with a laminar boundary layer. A zone of local supersonic flow ordinarily terminates in shock waves, whose form is influenced by the boundary layer. When it is laminar, a succes-

sion of so-called lambda shock waves first appears. This schlieren photograph with vertical knife edge shows a biconvex airfoil 12 per cent thick at free-stream Mach number 0.8. Photograph by H. W. Liepmann

(\*)



246. Single lambda shock wave on an airfoil with a laminar boundary layer. As the Mach number is increased the multiple shock waves of the upper photograph merge into one. Here a horizontal knife edge shows that at

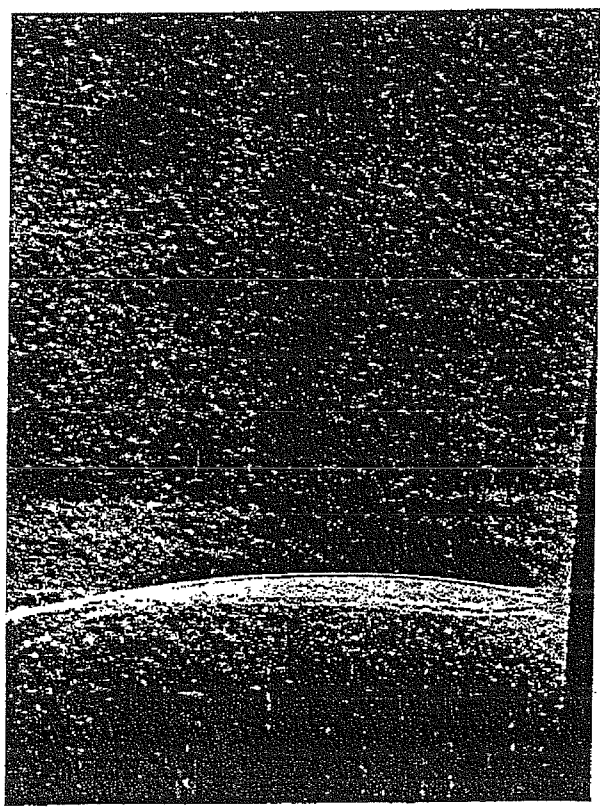
$M=0.9$  the laminar boundary layer is separated from the biconvex airfoil ahead of the shock wave and became turbulent. Photograph by H. W. Liepmann



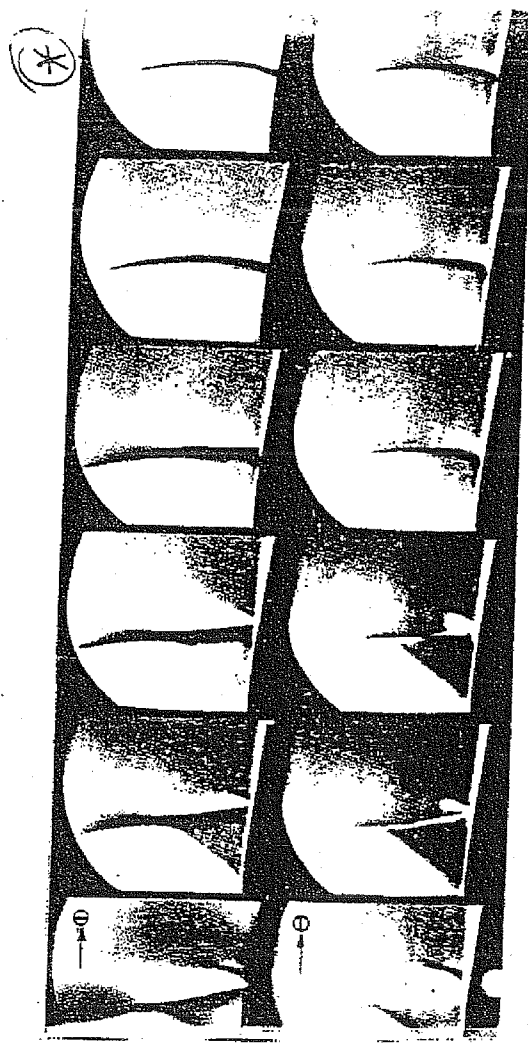
247. Shock waves on a laminar boundary layer with increasing Mach number. The lambda shock waves are seen to merge with increase of the local Mach number, whose value is given below each pair of schlieren photographs. Here the laminar boundary layer forms on a curved plate that produces a growing limited region of supersonic flow embedded in a subsonic flow, imitating supercritical flow over an airfoil. The symbols at the left indicate that the knife edge is vertical in the upper row and horizontal below. *Acleret, Feldmann & Rott 1946*

1.105 1.204 1.223 1.303

(\*)



248. Shock wave on an airfoil with a turbulent boundary layer. When the boundary layer is turbulent, a single shock wave appears. The flow then resembles the inviscid model more closely than when the layer is laminar. Here the free-stream Mach number is 0.84, and the knife edge is vertical to show the shock wave clearly. Photograph by H. W. Liepmann



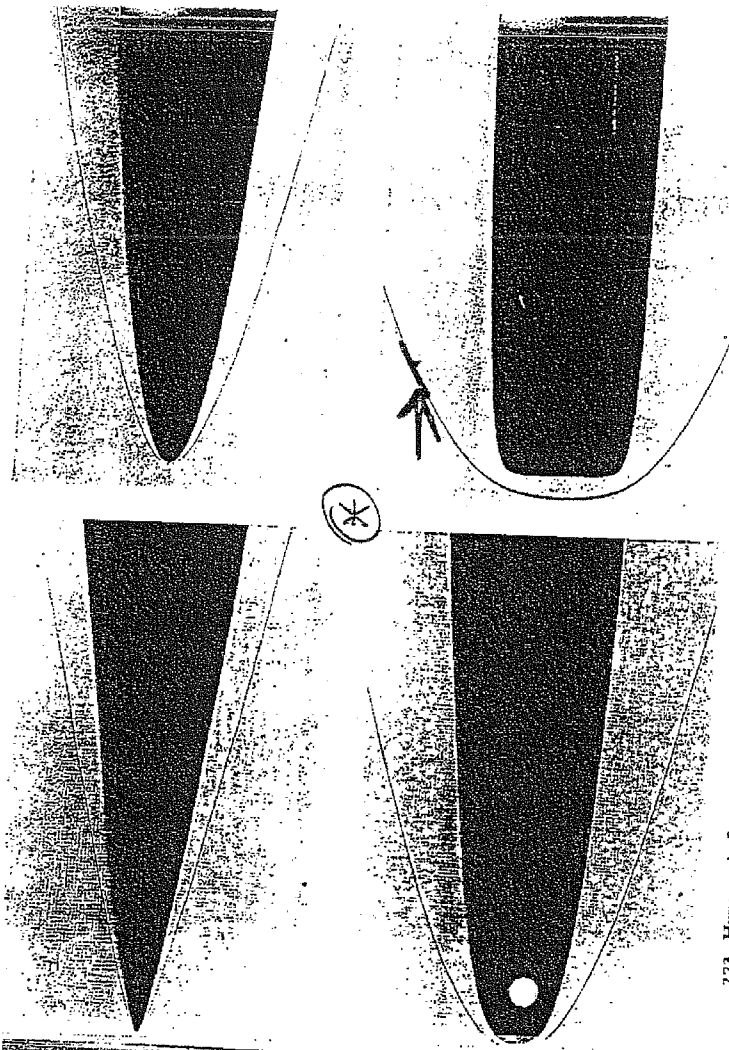
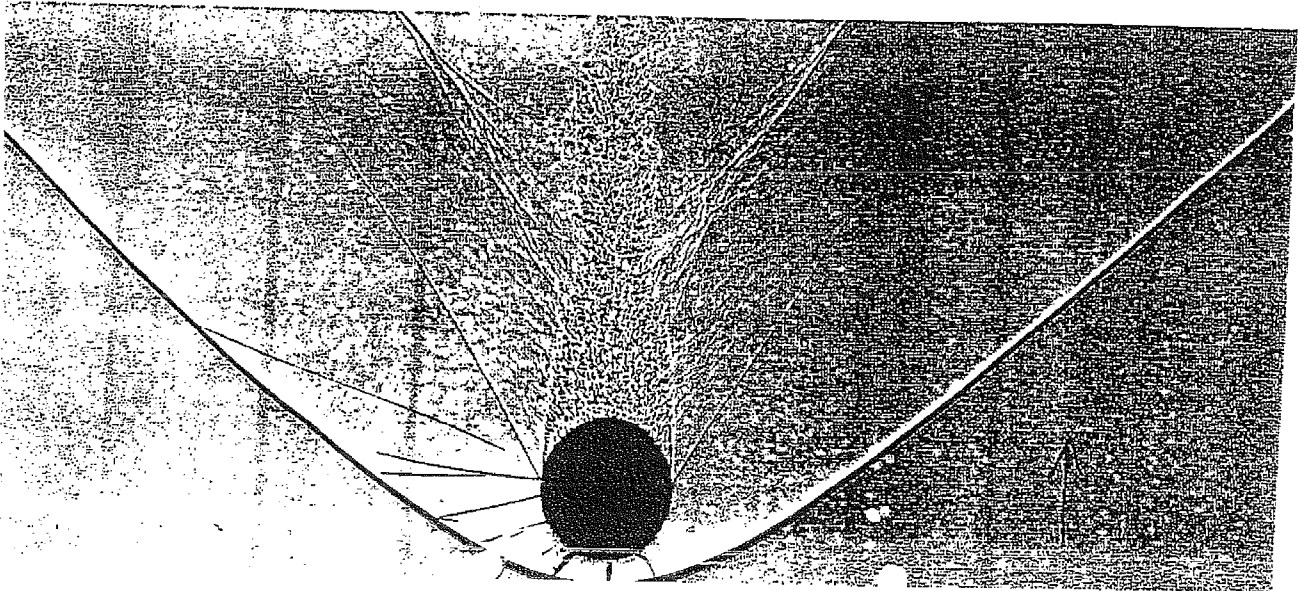
249. Shock waves on a laminar boundary layer becoming turbulent. The local Mach number on a curved plate remains almost fixed at 1.2 or 1.3 as the Reynolds number is doubled, progressing from 1,120,000 at the left to 2,660,000 at the right. As the boundary layer changes from laminar to turbulent ahead of the shock wave, the oblique leg of the lambda shock wave gradually disappears. *Acleret, Feldmann & Rott 1946*

(\*)



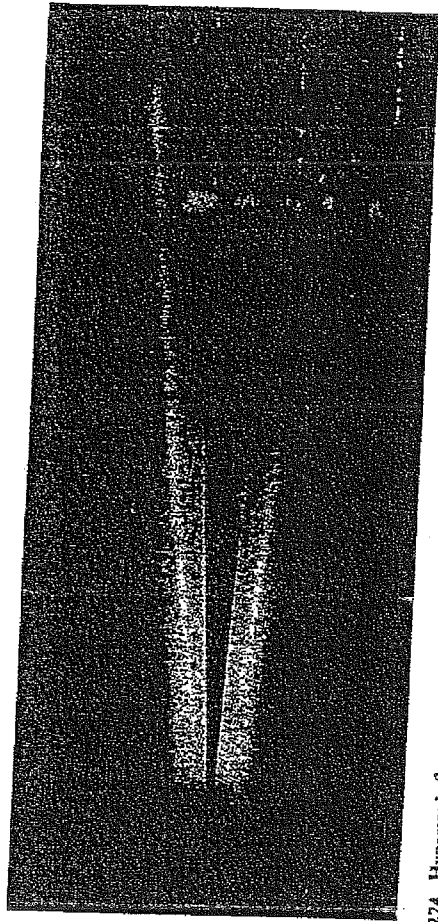
250. Shock wave in transonic flow over a bump. An infinite-ridge interferogram shows transonic flow over a 7-per-cent-thick circular-arc bump on a channel wall. The local region of supersonic flow terminates in a shock wave that interacts with the turbulent boundary layer on the wall, as in the preceding two photographs. *Déjery, Charrat & Le Bailleur 1975*

266. Sphere at  $M=1.53$ . A shadowgraph catches a 1/2-inch sphere in free flight through air. The flow is subsonic behind the part of the bow wave that is ahead of the sphere, and over its surface back to  $45^\circ$ . At about  $50^\circ$  the laminar boundary layer separates through an oblique shock wave, and quickly becomes turbulent. The fluctuating wake generates a system of weak disturbances that merge into the second shock wave. Photograph by A. C. Charters



273. Hypersonic flow past power-law bodies. Shadowgraphs show the bow wave in air at  $M=8.8$  for bodies of revolution whose radius varies as a power of axial distance.

The exponents are  $3/4$ ,  $1/2$  (a paraboloid of revolution),  $1/4$ , and  $1/10$ . Freeman, Cash & Redder 1964, courtesy of Aerodynamics Division, National Physical Laboratory

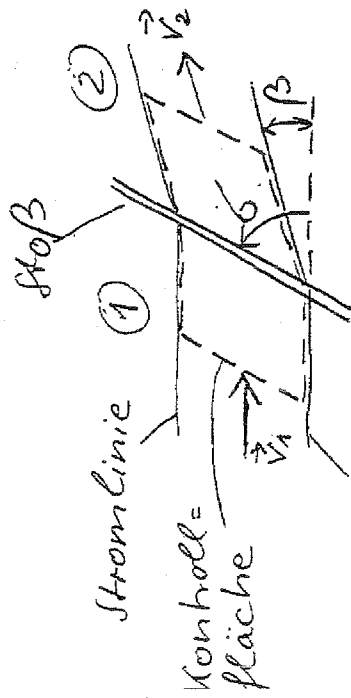


274. Hypersonic flow past a slender cone. A cone of  $3^\circ$  semi-vertex angle is shown by the glow-discharge method in helium at Mach number 41 and Reynolds number

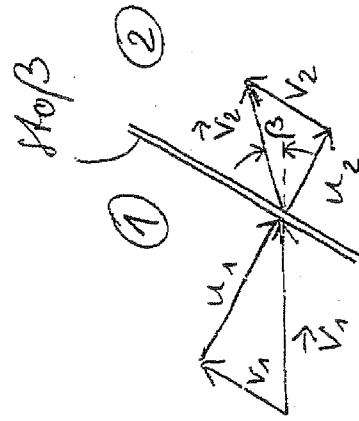
560,000 based on length. In this strong-interaction regime the boundary layer is seen to extend about four-fifths of the distance to the shock wave. Horvath & Kusny 1968

# Schräger Verdichtungsstoß

I. a. sind Verdichtungsstöße gegenüber d. Ausströmung geneigt  $\rightarrow$  d.  $\perp$  Verd. Stoß ist ein Sonderfall



Stromlinie



$\alpha$ : Stoßwinkel

$\beta$ : Umkehrwinkel

Formulierung d. Erhaltungsgleichungen über den Stoß

Kontinuität:

$$S_1 u_1 = S_2 u_2$$

Imp. satz, tangential:

$$S_1 u_1 v_1 = S_2 u_2 v_2$$

Imp. satz, normal:

$$S_2 u_2^2 - S_1 u_1^2 = P_2 - P_1$$

Energiesatz

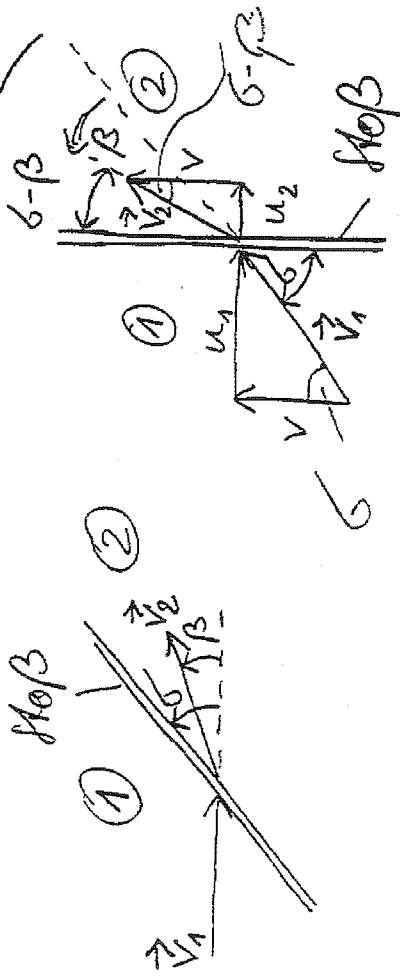
$$\frac{\|\vec{v}_2\|^2}{2} - \frac{\|\vec{v}_1\|^2}{2} = c_p(T_1 - T_2)$$

Imp. satz, tangential:

$$\Rightarrow v_1 = v_2 = v_0$$



⇒ Umkehrung von v-Feld mögl.



⇒ Analyse wie beim

⊥ Vrel. Stoß :

$$\|\vec{V}_1\| = \sqrt{u_1^2 + v^2}$$

$$, \quad \delta = \tan^{-1}(u_1/v)$$

$$\|\vec{V}_2\| = \sqrt{u_2^2 + v^2}$$

$$, \quad \delta - \beta = \tan^{-1}(u_2/v)$$

!  $u_2 < u_1 \Rightarrow$  Umkehrung oder  
Stromung im  
Richtung des Stoßes!

Mach Zahlen ⊥ zum Stoß

$$M_{n1} = u_1 / c_1 = M_1 \sin \delta > 1$$

$$M_{n2} = u_2 / c_2 = M_2 \sin(\delta - \beta) <$$

$$P_2/P_1, \quad S_2/S_1, \quad T_2/T_1, \quad \frac{\Delta S}{R} \text{ durch}$$

$$M_1 \leftarrow M_{n1} = M_1 \sin \delta$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} (M_1^2 \sin^2 \delta - 1)$$

$$\frac{S_2}{S_1} = \frac{u_1}{u_2} = \frac{(\gamma+1) M_1^2 \sin^2 \delta}{(\gamma-1) M_1^2 \sin^2 \delta + 2} = \frac{\tan \delta}{\tan(\delta-1)}$$

$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2} \frac{\gamma M_1^2 \sin^2 \delta + 1}{M_1^2 \sin^2 \delta} (M_1^2 \sin^2 \delta - 1)$$

$$\frac{S_2 - S_1}{R} = \ln \left\{ \left[ \frac{(\gamma+1) M_1^2 \sin^2 \delta}{(\gamma-1) M_1^2 \sin^2 \delta + 2} \right]^{\frac{\gamma}{\gamma-1}} \right.$$

$$\left. \left[ 1 + \frac{2\gamma}{\gamma+1} (M_1^2 \sin^2 \delta - 1) \right]^{\frac{1}{\gamma-1}} \right\}$$

die Prandtl Beziehung d. 1. Verd. St.

$$u_1 u_2 = c^{*2}$$

nimmt f. d. schrägen Verd. St. folgende Form an:

Energiesatz

$$c_p T_1 + \frac{u_1^2 + v^2}{2} = c_p T_2 + \frac{u_2^2 + v^2}{2} = c_p T_0$$

mit  $c_p = \gamma R / (\gamma - 1)$

$$\frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{u_1^2 + v^2}{2} = \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{u_2^2 + v^2}{2} = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}$$

Ruhe- und krit. Größen

$$c^{*2} \frac{\gamma + 1}{2(\gamma - 1)} = c_0^2 \frac{1}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}$$

⇒

$$p_1 = \rho_1 \left[ \frac{\gamma + 1}{2\gamma} c^{*2} - \frac{\gamma - 1}{2\gamma} (u_1^2 + v^2) \right]$$

$$p_2 = \rho_2 \left[ \frac{\gamma + 1}{2\gamma} c^{*2} - \frac{\gamma - 1}{2\gamma} (u_2^2 + v^2) \right]$$

Im Imp. Satz, nimmt

$$S_1 \left[ \frac{\gamma + 1}{2\gamma} (u_1^2 + c^{*2}) - \frac{\gamma - 1}{2\gamma} v^2 \right] =$$

$$S_2 \left[ \frac{\gamma + 1}{2\gamma} (u_2^2 + c^{*2}) - \frac{\gamma - 1}{2\gamma} v^2 \right]$$

+ Konti. glg. ⇒ Prandtl Beziehung.

für d. schrägen Verd. Stoß

$$u_2 u_1 = c^{*2} - \frac{\gamma - 1}{\gamma + 1} v^2$$

$M_2 = f(M_1, \delta, \beta, \gamma)$  aus  $M_2 = g(M_1, \gamma)$  mittels  $M_1 \leftarrow M_1 \sin \delta$  und

$$M_2 \leftarrow M_2 \sin(\delta - \beta)$$

$$M_2^2 \sin^2(\delta - \beta) = \frac{(\gamma - 1) M_1^2 \sin^2 \delta + 2}{2\gamma M_1^2 \sin^2 \delta - (\gamma - 1)}$$

Zusammenhang zwischen

$\beta, \sigma, M_1$  mittels  $s_{\frac{1}{2}} = f(M_1, \sigma) = g(\sigma, \beta)$

mit  $\tan(\sigma - \beta) = \frac{\tan \sigma - \tan \beta}{1 + \tan \sigma \tan \beta}$

$$\Rightarrow \tan \beta = \frac{2 \cot \sigma \frac{M_1^2 \sin^2 \sigma - 1}{M_1^2 (\gamma + \cos 2\sigma)} + 2}{1}$$

$$\sigma = \frac{\pi}{2} \rightarrow \tan \beta = 0 \Rightarrow \text{Max. } \rho_{\text{ex. } 0}$$

$$\sigma = \sigma_1 = \sin^{-1} \left( \frac{1}{M_1} \right) \rightarrow \tan \beta = 0$$

Analyse für  $M_1 \rightarrow \infty$ :

$$r = \lim_{M_1 \rightarrow \infty} (\tan \beta) = \frac{2 \cot \sigma \sin^2 \sigma}{\gamma + \cos 2\sigma} = \frac{\sin 2\sigma}{\gamma + \cos 2\sigma}$$

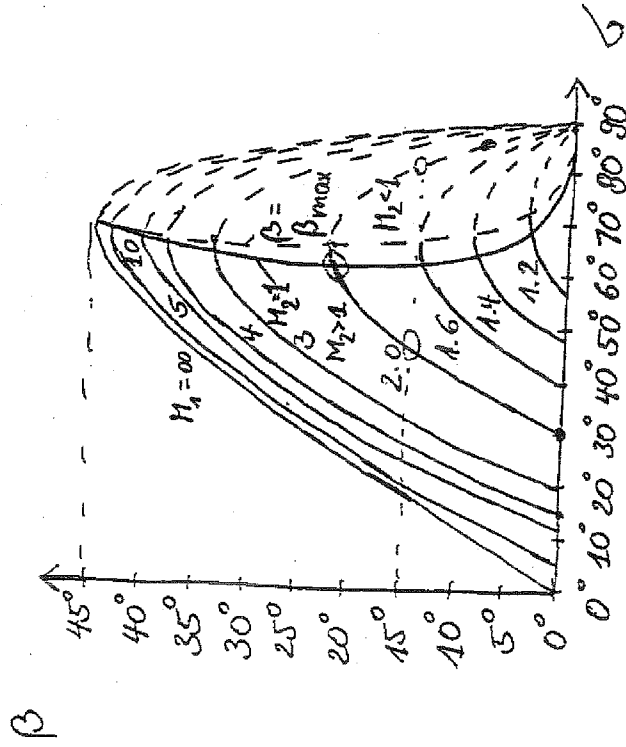
$$\frac{dr}{d\sigma} = \frac{2 \cos 2\sigma (\gamma + \cos 2\sigma) - 2 \sin^2 2\sigma}{(\gamma + \cos 2\sigma)^2} \stackrel{!}{=} 0$$

$$\Rightarrow \cos 2\sigma = -\frac{1}{\gamma} ; \gamma = 1.4 : \sigma = 67.5$$

$$\tan \beta_{\text{max}} = \frac{\sin 2\sigma}{\gamma + \cos 2\sigma} = \frac{\sqrt{1 - \cos^2 2\sigma}}{\gamma + \cos 2\sigma} =$$

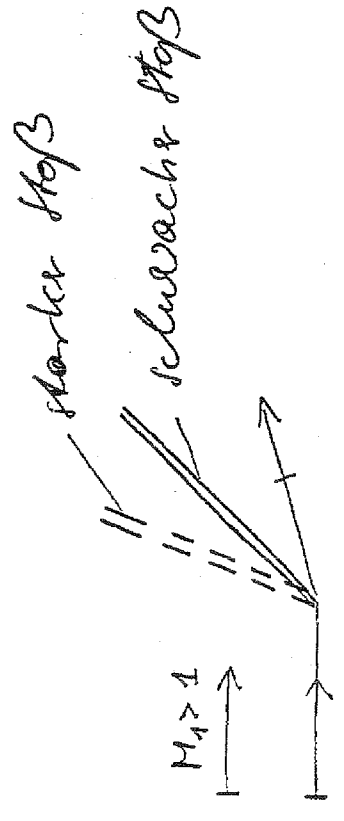
$$\frac{\sqrt{1 - \frac{1}{\gamma^2}}}{\gamma - \frac{1}{\gamma}} = \frac{\sqrt{\gamma^2 - 1}}{\gamma^2 - 1} = (\gamma^2 - 1)^{-\frac{1}{2}}$$

$$\beta_{\text{max}} \approx 45^\circ$$



$\beta < \beta_{max}$  : 2 Lsgn.

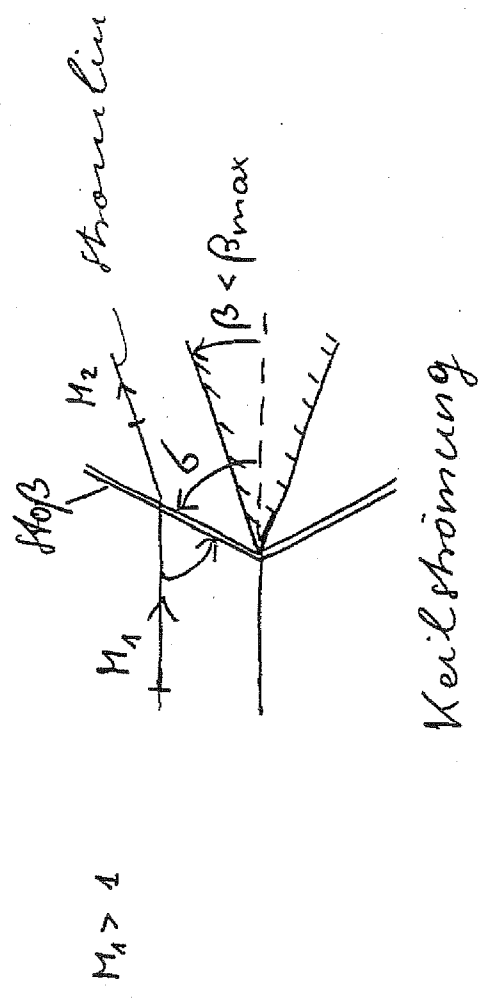
- schwache Lsg.,  $M_2 > 1$  (i.a.)
- starke Lsg.,  $M_2 < 1$



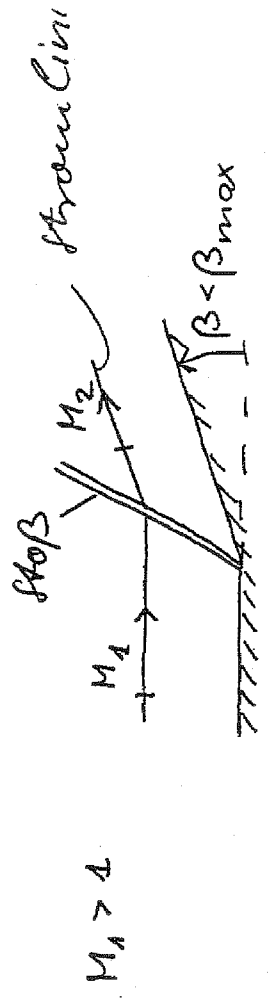
schwache Lsg.  $\hat{=}$  d. natürl. Lsg.

$\beta > \beta_{max}$  : keine geschlossene, analytische Lösung

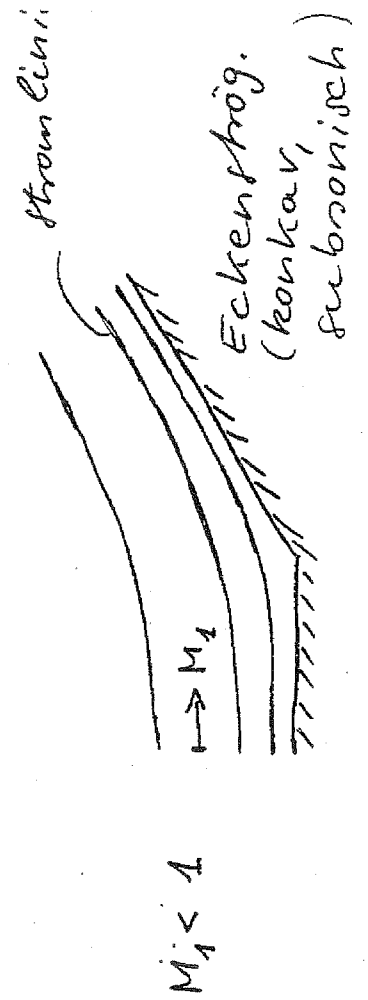
Entstehung von schrägen VdL-St.



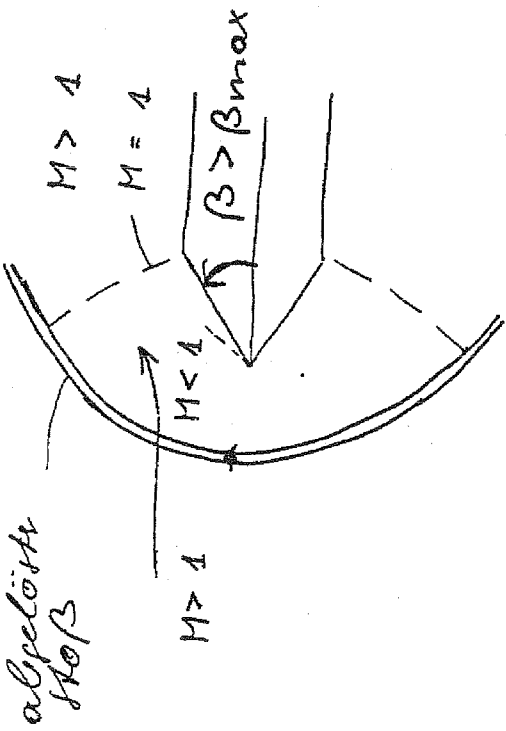
Keilströmung



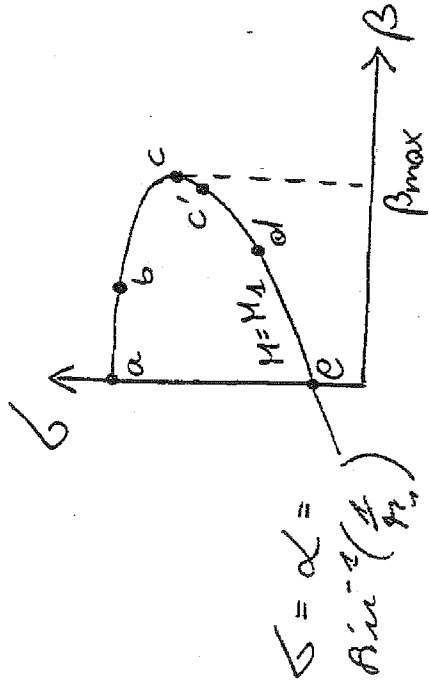
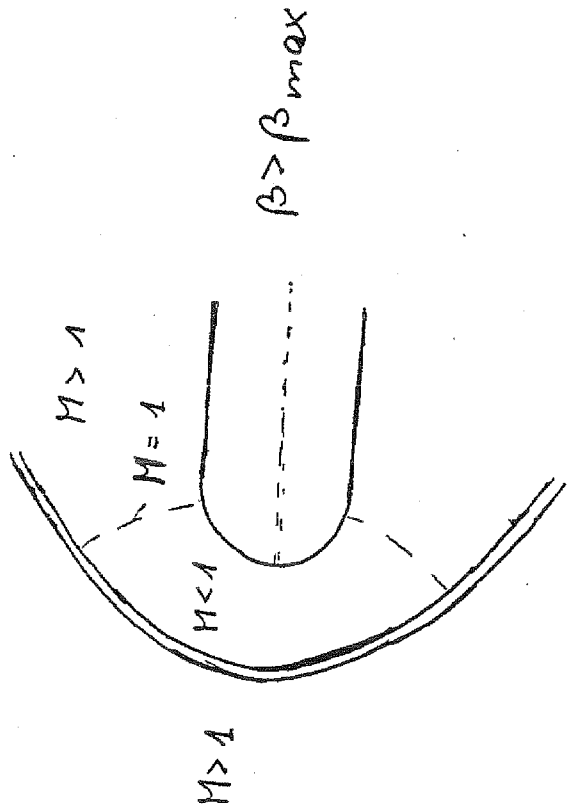
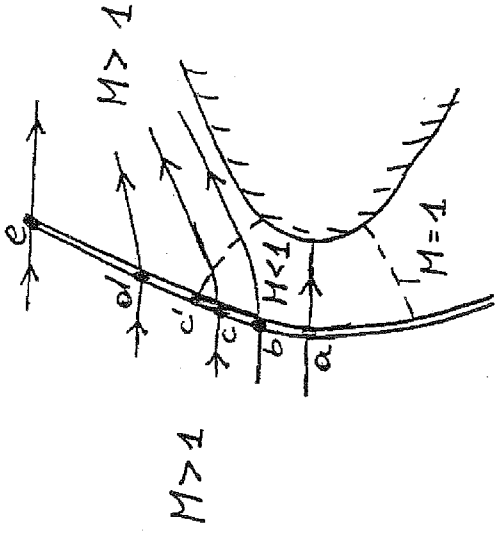
Eckenströmung (konkav)



$\beta > \beta_{max}$  : Ablösg. od. Verd. Stoß



abgelöste  
Verd. Stoß



$\beta, \beta, M_1$  - 2. Stg.

## Approximationen für schwache von. St.

• Hyperschallströmungen:

$$M_1^2 \sin^2 \delta \gg 1, \quad \sin \delta \ll 1$$

$$\text{aus } \frac{\tan(\delta - \beta)}{\tan \delta} = \frac{\gamma - 1 + 2/M_1^2 \sin^2 \delta}{\gamma + 1}$$

$$\text{und } \delta \approx \beta$$

$$\Rightarrow \frac{\delta - \beta}{\delta} = \frac{\gamma - 1}{\gamma + 1}$$

$$\delta = \frac{\gamma + 1}{2} \beta \quad \text{Newton-} \\ \text{Theorie}$$

• Näherung für schwache Stöße:

Annahme:  $\beta$  klein

$$\beta \rightarrow 0: \Rightarrow \delta \rightarrow \alpha_1 = \sin^{-1}(1/M_1)$$

$$\text{w. } \beta = f(M_1, \delta)$$

$$\Rightarrow \tan \beta \approx \frac{M_1^2 \sin^2 \delta - 1}{M_1^2 (\gamma + \cos 2\alpha_1) + 2}$$

mit

$$\tan \beta \approx \beta$$

$$\cot \alpha_1 = \frac{\cos \alpha_1}{\sin \alpha_1} = \frac{\sqrt{1 - \sin^2 \alpha_1}}{\sin \alpha_1} = \sqrt{M_1^2 - 1}$$

$$\cos 2\alpha_1 = 1 - 2 \sin^2 \alpha_1 = 1 - 2/M_1^2$$

$$\Rightarrow M_1^2 \sin^2 \delta - 1 \approx \frac{M_1^2 (\gamma + 1)}{2 \sqrt{M_1^2 - 1}} \beta$$

$$\text{in } P_2/P_1 = h(M_1, \delta)$$

$\Rightarrow$  relative Druckänderg. für  $\beta$  klein

$$\frac{P_2 - P_1}{P_1} \approx \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}} \beta$$

gültig: schwache Komp. + Exp. wellen

Grund:

$$\frac{\Delta S}{R} \approx \frac{\gamma + 1}{12 \gamma^2} \left( \frac{\Delta P}{P_1} \right)^3 = \frac{\gamma + 1}{12 \gamma^2} \left( \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}} \right)^3 \beta^3$$

$\Rightarrow$  schw. Stöße näherungsweise isentrop  $\rightarrow$  reversibel